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SOME EXTENSIONS OF 'RAYLEIGH'S PRINCIPLE'

By R. V. SOUTHWELL (*Cambridge*)

[Received 4 December 1952]

SUMMARY

Retaining Lord Rayleigh's postulate, that any permitted displacement can be expressed in a series of 'normal modes', this paper waives his restriction to systems of which the eigenvalues all have one sign. A simple modification of his approximate expression for an eigenvalue allows the wider class of problems to be treated similarly; and used in conjunction with 'intensification' (the term used here for the iterative process due to Schwarz, Vianello, and/or Stodola) it permits an imposition of *double* limits, also yields estimates of other (unwanted) components in an approximation to the gravest mode.

This last feature will have value—in problems which cannot be solved exactly—if two or more of the gravest eigenvalues are nearly or exactly equal. Brief reference is made to an example of that kind, relating to the elastic stability of a flat-plate cantilever. It will be treated in a separate paper concerned with approximate computation of *modes* of 'buckling' (collapse).

Introduction

LORD RAYLEIGH's presentation (**1**) of his well-known 'principle' relates to problems in which the eigenvalues (squares of the natural frequencies) are all positive; and strictly speaking assumes the freedom to be finite, since it postulates the validity of expansion in terms of the eigenfunctions ('normal modes'). This paper, waiving the restriction to one-signed eigenvalues, retains his postulate regarding such expansion. Rigorous proof, as entailing particular scrutiny of each equation, is not compatible with its purpose, which is generalization. Moreover, *in effect* the freedom is made finite in numerical treatment (e.g. relaxation), when derivatives are replaced by finite differences.

The particular class of problems which the paper contemplates is concerned with the stability of flat plates sustaining edge-thrusts: in it, the eigenvalues define the 'critical' intensities of that loading, the normal modes are the modes of flexural displacement (w) which the critical loadings can maintain. When small, w is governed by an equation of the type

$$V(w) = \lambda\, T(w),$$

in which V and T are symmetric operators, *not* necessarily positive. One of the worked examples is a problem of this kind.

But the *methods* have much wider application, and three extensions of Rayleigh's method are believed to be new, viz.

(i) 'Vaisey's theorem' (section 4),

(ii) an alternative (μ_S) to 'Rayleigh's estimate' λ_R, obtained by a use of one 'intensification' (section 7),

(iii) two forms of 'restrictive inequality' (sections 11 and 13) whereby *double* limits are set to the numerically smallest eigenvalue (λ_1) or to another eigenvalue nearly equal to λ_1.

The double limits have value as putting definite bounds to the error of a 'Rayleigh estimate': (1) because when the error is acceptably small computation can be stopped, and (2) because, when λ_1 has been determined closely, more can be learned about the associated mode w_1.

1. For a system defined by a finite number of coordinates, the equations which govern a normal mode associated with an eigenvalue λ have forms typified by

$$\partial \mathbf{V}/\partial a_k - \lambda\, \partial \mathbf{T}/\partial a_k = 0 \quad (k = 1, 2, \ldots, N), \tag{1}$$

in which a_k is a representative coordinate and $\mathbf{V}$ and $\mathbf{T}$ are homogeneous quadratic functions of the a's. For a continuous system ($N = \infty$) the N equations (1) are replaced by a differential equation of the form

$$\vartheta w - \lambda\, \vartheta' w = 0, \tag{2}$$

which holds at every point. In this paper ϑ and ϑ' will denote *any* self-adjoint linear operators such that

$$I \text{ (say)} \equiv \ldots \iint w\, \vartheta w\, dxdy \ldots, \qquad I'/\Lambda \text{ (say)} \equiv \ldots \iint w\, \vartheta' w\, dxdy \ldots$$

are homogeneous quadratic functions of the 'displacement' w, corresponding with $\mathbf{V}$ and $\mathbf{T}$, respectively, in (1); and whereas in Lord Rayleigh's problems both $\mathbf{V}$ and $\mathbf{T}$ were essentially positive, *here only I is required to have that property, the other integral I'/Λ may take either sign.* In consequence the eigenvalues can have both signs, but must still (cf. section 2) be real: those which are positive will be denoted by

$$\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots,$$

those which are negative by

$$\lambda_{-1}, \lambda_{-2}, \ldots, \lambda_{-l}, \ldots,$$

both series ascending in order of numerical magnitude. (The significance of Λ will appear in section 5.)

With (2) are associated 'boundary conditions' which express the operation of restraints. The number of the independent variables $x, y, \ldots$, is not restricted, but in practical applications is normally one or two, and is taken as two in this presentation of fundamental theory. *For brevity the same symbols will be employed both for ϑw, $\vartheta' w$, $\iint w\, \vartheta w\, dxdy$ and for their approximations in terms of finite differences.*

2. Since (2), being linear, does not define completely the amplitudes of the normal modes (eigenfunctions), we 'normalize' them by the additional requirement

$$\iint w_k\, \vartheta w_k\, dxdy = 1 = \lambda_k \iint w_k\, \vartheta' w_k\, dxdy \tag{3}$$

(which implies no additional limitation, and is consistent with the postulate that I is positive). In virtue of the conditions of restraint, any two of them satisfy 'conjugate relations' of the types

$$\left.\begin{aligned} \iint w_r\,\vartheta w_s\,dxdy = 0 = \iint w_s\,\vartheta w_r\,dxdy \\ \iint w_r\,\vartheta' w_s\,dxdy = 0 = \iint w_s\,\vartheta' w_r\,dxdy \end{aligned}\right\} (\lambda_r \neq \lambda_s). \tag{4}$$

These relations serve to establish the reality of the eigenvalues. For the existence of a complex eigenvalue $A+iB$, associated with a complex mode $W+iW'$ (say), would imply the existence of an eigenvalue $A-iB$ associated with a mode $W-iW'$; and when these modes are identified with w_r and w_s, unless $B = 0$ (so that $\lambda_r \equiv \lambda_s$) we have, according to the first of (4),

$$\iint (W\,\vartheta W + W'\,\vartheta W')\,dxdy = 0,$$

therefore $W \equiv 0 \equiv W'$ (since I is essentially positive), whether all of the eigenvalues have one sign or not.

Rayleigh's principle

3. The postulated validity of expansion in normal modes (section 1) requires that, when the A's have appropriate values, the series

$$\begin{aligned} w &= A_1 w_1 + A_2 w_2 + \ldots + A_k w_k + \ldots \\ &= \sum (A_k w_k), \text{ say,}\dagger \end{aligned} \tag{5}$$

will represent any 'mode of displacement' (w) which does not violate the conditions of restraint, *and may be differentiated*. From (5), in virtue of (3) and (4), it follows that

$$\left.\begin{aligned} I &\equiv \iint w\,\vartheta w\,dxdy = \sum (A_k^2) \\ I' &\equiv \Lambda \iint w\,\vartheta' w\,dxdy = \Lambda \sum (A_k^2/\lambda_k) \end{aligned}\right\}, \tag{6}$$

consequently λ_R as defined by

$$\lambda_R \equiv \Lambda I/I' \equiv \sum (A_k^2)/\sum (A_k^2/\lambda_k) \tag{7}$$

has a stationary value when the mode w is 'normal' so that all of the A's are zero except λ_s (say), because then $\lambda_R = \lambda_s$, so

$$\begin{aligned} \tfrac{1}{2}(I'/\Lambda)\,\partial\lambda_R/\partial A_k = A_k(1-\Lambda I/\lambda_k I') &= A_k(1-\lambda_R/\lambda_k) \\ &= 0 \quad \textit{for every } k. \end{aligned}$$

This is 'Rayleigh's principle', which (cf. section 1) he propounded as holding for systems whose eigenvalues are all positive. It holds also when, as here, the eigenvalues have both signs but are all real.

† A normal mode will be termed 'positive' or 'negative' in conformity with the sign of its associated λ. $\sum_k$ will denote a summation extending only to positive modes, typified by w_k; $\sum_{-l}$ a summation extending only to negative modes, typified by w_{-l}; $\sum$ a summation extending to every normal mode.

Vaisey's theorem

4. Lord Rayleigh (**1**) argued, further, that λ_R as defined in (7) always has a value intermediate between the least and the greatest of them (or above the least, when the system has infinite freedom), consequently λ_R as computed for a guessed form w (here termed the 'Rayleigh estimate') may be regarded as an *upper limit* to λ_1. This theorem fails when I'/Λ can take either sign or vanish, for in the last event λ_R has an infinite value. But it is easy to establish a theorem—due to Miss G. Vaisey—*that λ_R cannot lie between the least positive and least negative eigenvalue* (λ_1 and λ_{-1}).

If λ_k and λ_{-l} typify the positive and the negative eigenvalues (cf. section 1), forms equivalent to (6) are

$$\left.\begin{aligned} I &= \sum_k (A_k^2) + \sum_{-l} (A_{-l}^2) = \mathbf{P}+\mathbf{Q}, \text{ say,} \\ I'/\Lambda &= \sum_k (A_k^2/\lambda_k) + \sum_{-l} (A_{-l}^2/\lambda_{-l}) = \mathbf{R}-\mathbf{S}, \text{ say,} \\ &\text{and as defined in (7)} \\ \lambda_R &\equiv \Lambda I/I' = (\mathbf{P}+\mathbf{Q})/(\mathbf{R}-\mathbf{S}), \end{aligned}\right\} \tag{8}$$

where **P**, **Q**, **R**, **S** are essentially positive quantities. Lord Rayleigh's argument can be employed to prove that

$$\mathbf{P} \geqslant \lambda_1 \mathbf{R}, \qquad \mathbf{Q} \geqslant |\lambda_{-1}|\mathbf{S} = -\lambda_{-1}\mathbf{S},$$

and it follows that

$$\begin{aligned} &(\mathbf{R}-\mathbf{S})^2(\lambda_R-\lambda_1)(\lambda_R-\lambda_{-1}) \\ &\qquad = \{(\mathbf{P}-\lambda_1\mathbf{R})+\mathbf{Q}+\lambda_1\mathbf{S}\}\{\mathbf{P}+|\lambda_{-1}|\mathbf{R}+\mathbf{Q}-|\lambda_{-1}|\mathbf{S}\} \\ &\qquad > 0. \end{aligned}$$

This establishes the theorem.

'Intensification'

5. The practical importance of Rayleigh's principle lies in his use of it (section 4) to obtain an upper limit to λ_1 by simple computations based on a *guess* regarding the form of w_1. In most problems of the sort which he considered the form can be guessed with considerable accuracy, and then in virtue of the stationary property (section 3) the resulting estimate of λ_1 is extremely close. But in the wider class considered here little confidence can be put in a guess regarding w_1: then, it may be improved by a process, here termed 'intensification', which is variously attributed to Schwarz (**2**), Vianello (**3**), and Stodola (**4**). This derives from w, successively, other forms w_{I}, w_{II},..., etc., satisfying the edge-conditions and

$$\left.\begin{aligned} \vartheta w_{\mathrm{I}} &= \Lambda\vartheta' w \\ \vartheta w_{\mathrm{II}} &= \Lambda\vartheta' w_{\mathrm{I}} \\ &\text{..., etc.,} \end{aligned}\right\}, \tag{9}$$

in which Λ is any convenient multiplier.

Provided that w_1 is a component of the guessed form w, when every eigenvalue is positive, then w_{I}, w_{II},..., etc., will approximate more and more closely to w_1; when the eigenvalues have both signs they will approximate to w_1 or w_{-1} according as λ_1 or $|\lambda_{-1}|$ is the smaller. For if w is expressible in the series

$$w = \sum (A_k w_k), \tag{5) bis$$

then

$$\left.\begin{aligned} w_{\mathrm{I}} &= \Lambda \sum (A_k w_k/\lambda_k) \\ w_{\mathrm{II}} &= \Lambda^2 \sum (A_k w_k/\lambda_k^2) \\ &\ldots, \text{etc.} \end{aligned}\right\}, \tag{10}$$

so the gravest mode becomes successively more predominant in w_{I}, w_{II},..., etc.

'Intensification' used in conjunction with Rayleigh's principle

6. Temple (**5**) and Temple and Bickley (**6**) have used intensification in conjunction with Rayleigh's principle; and Koch (**7**) did the same thing *in effect*, by propounding as an upper limit to λ_1

$$\Lambda \iint w\, \vartheta w\, dxdy \Big/ \iint w\, \vartheta w_{\mathrm{I}}\, dxdy \equiv \iint w\, \vartheta w\, dxdy \Big/ \iint w\, \vartheta' w\, dxdy$$

(in the notation of this paper), for the ratio on the right is λ_R. By this procedure, *when the eigenvalues are all positive*, the accuracy of Rayleigh's estimate is improved: for, as derived from w, it is given by (**7**), and as derived from w_{I} it is

$$\left.\begin{aligned} (\lambda_R)_{\mathrm{I}} &\equiv \Lambda\, I''/I''', \text{ say}, \\ \text{where} \qquad I'' &\equiv \Lambda^2 \iint w_{\mathrm{I}}\, \vartheta w_{\mathrm{I}}\, dxdy = \Lambda^2 \sum (A_k^2/\lambda_k^2) \\ I''' &\equiv \Lambda^3 \iint w_{\mathrm{I}}\, \vartheta w_{\mathrm{II}}\, dxdy = \Lambda^3 \sum (A_k^2/\lambda_k^3) \end{aligned}\right\}, \tag{11}$$

so

$$\begin{aligned} (\lambda_R)_{\mathrm{I}}/\lambda_R &= \sum (A_k^2/\lambda_k^2) \sum (A_k^2/\lambda_k)/\{\sum (A_k^2/\lambda_k^3) \sum (A_k^2)\} \\ &= 1 - \sum_{r,s} [A_r^2 A_s^2 (\lambda_r^{-1} - \lambda_s^{-1})^2 (\lambda_r^{-1} + \lambda_s^{-1})]/\sum (A_k^2/\lambda_k^3) \sum (A_k^2) \end{aligned} \tag{12}$$

$\Big(\sum_{r,s}$ extending to all pairs such as r and $s\Big)$

$\leqslant 1$ *when every λ is positive* (otherwise the co-factor of some $A_r^2 A_s^2$, in (12), will be negative).

An alternative to the 'Rayleigh estimate'

7. All of Rayleigh's arguments apply to

$$\mu_S^2 \text{ (say)} \equiv \Lambda^2 I/I'' \equiv \sum (A_k^2)/\sum (A_k^2/\lambda_k^2), \tag{13}$$

whether all of the eigenvalues have one sign or not, because every $\lambda^2 > 0$. That is to say,

$$\mu_S^2 \geqslant \text{the smaller of } \lambda_1^2 \text{ and } \lambda_{-1}^2,$$

also, as in (12),

$$(\mu_S^2)_{\mathrm{I}} \leqslant \mu_S^2, \text{ when } (\mu_S^2)_{\mathrm{I}} \text{ is derived from } w_{\mathrm{I}}.$$

Finally, by (7) and (13),

$$\mu_S^2/(\lambda_R)^2 \equiv I'^2/I\,.\,I'' = [\sum (A_k^2/\lambda_k)]^2/\sum (A_k^2/\lambda_k^2)\,.\,\sum (A_k^2)$$
$$= 1 - \sum_{r,s} [A_r^2 A_s^2(\lambda_r^{-1}-\lambda_s^{-1})^2]/\sum (A_k^2/\lambda_k^2)\,.\,\sum (A_k^2) \quad (14)$$
$$\leqslant 1, \textit{ whatever the sign of the eigenvalues.}$$

Thus for every system intensification should invariably reduce $|\mu_S|$, an estimate which imposes a closer limit than λ_R on the magnitude of the numerically smallest eigenvalue. Consequently *a failure of* $|\mu_S|$ *to decrease is unambiguous evidence of a computational error.* Vaisey's theorem (section 4) is not required.

A price in labour is paid for these advantages, since w *must* be intensified (to obtain w_{I}) before a value can be attached to I''; but it is justified by the resulting information, in the complex problems which this paper contemplates. The indication of error is especially valuable, since it shows when the attainable accuracy is limited: in theory, intensification cannot fail (by reducing unwanted components) to improve any guess to which it is applied; but an argument which neglects inescapable errors cannot yield a trustworthy assessment of its *practical* merits. Much has been written which presumes exact solution to be impracticable, yet proceeds as though such errors were avoidable.

Rayleigh's principle used in conjunction with two intensifications

8. Advantage having accrued from one intensification, it is natural to inquire whether two or more will be justified by the resulting gain in information. Two at least should normally be practicable, and their cumulative error should not be serious: i.e. they should yield close approximations to w_{I}, w_{II} and values for the integrals

$$\left.\begin{aligned}
I &\equiv \iint w\, \vartheta w\, dxdy &&= \sum [A_k^2] \\
I' &\equiv \iint w\, \vartheta w_{\mathrm{I}}\, dxdy &&= \Lambda \sum [A_k^2/\lambda_k] \\
I'' &\equiv \iint w\, \vartheta w_{\mathrm{II}}\, dxdy &&= \Lambda^2 \sum [A_k^2/\lambda_k^2] \equiv \iint w_{\mathrm{I}}\, \vartheta w_{\mathrm{I}}\, dxdy \\
I''' &\equiv \iint w_{\mathrm{I}}\, \vartheta w_{\mathrm{II}}\, dxdy &&= \Lambda^3 \sum [A_k^2/\lambda_k^3] \\
I^{\mathrm{iv}} &\equiv \iint w_{\mathrm{II}}\, \vartheta w_{\mathrm{II}}\, dxdy &&= \Lambda^4 \sum [A_k^2/\lambda_k^4]
\end{aligned}\right\}, \quad (15)$$

of which I, I', I'' have appeared already, in sections 3 and 6. Taking these values as reliable, the remainder of this paper draws conclusions in regard to the modes and eigenvalues.

The argument is based throughout on Rayleigh's principle, occasionally extended in the manner of sections 4, 6, and 7. In sections 18–20 it will

be applied to a representative set of computed values: here it is illustrated by an example that can be solved exactly—a uniform strut with hinges at both ends.† For that system (w denoting the transverse displacement)

$$\vartheta w \equiv -d^2w/dx^2, \qquad \vartheta' w \equiv w, \qquad \lambda \equiv PL^2/B,$$

when B denotes the uniform flexural rigidity, L the length, and P the thrust, and when Lx measures the distance from one end. The eigenvalues—all positive—are

$$\lambda_1, \lambda_2, \lambda_3,\ldots, \text{etc.} = \pi^2\times(1, 2^2, 3^2,\ldots, \text{etc.}), \tag{16}$$

and (9) take the forms

$$d^2w_{\mathrm{I}}/dx^2+\Lambda w = 0, \qquad d^2w_{\mathrm{II}}/dx^2+\Lambda w_{\mathrm{I}} = 0,\ldots, \text{etc.},$$

$$\left.\begin{aligned} &\text{so when the guessed mode is } w = x(1-x), \\ &\text{then} \qquad 12w_{\mathrm{I}} = \Lambda(x-2x^3+x^4), \\ &\qquad 360w_{\mathrm{II}} = \Lambda^2(3x-5x^3+3x^5-x^6), \\ &\text{and} \qquad 3I = 1, \quad 30I' = \Lambda, \quad 5040I'' = 17\Lambda^2, \\ &144\times 630I''' = 31\Lambda^3, \quad 1386\times 14400I^{\mathrm{iv}} = 691\Lambda^4. \end{aligned}\right\} \tag{17}$$

Therefore in this example, according to (7),

λ_R as deduced from w is $\Lambda I/I' = 10$,

λ_R as deduced from w_{I} is $\Lambda I''/I''' = 17\times 18/31 = 9{\cdot}870\,967\,74$,

according to (13),

μ_S^2 as deduced from w is $\Lambda^2 I/I'' = 1680/17 = (9{\cdot}941\,002\,434)^2$,

μ_S^2 as deduced from w_{I} is $\Lambda^2 I''/I^{\mathrm{iv}} = 34\times 1980/691$

$= (9{\cdot}870\,360\,84)^2$.

Observing that for this system, according to (16),

$$\lambda_1 = \pi^2 = 9{\cdot}869\,604\,401\,089\,359, \tag{18}$$

we see that

$$\lambda_R > (\lambda_R)_{\mathrm{I}} > \lambda_1, \quad \text{as predicted in section 6,}$$

and that $\qquad \mu_S^2 > (\mu_S^2)_{\mathrm{I}} > \lambda_1^2,$

also $\qquad \mu_S^2 < \lambda_R^2, \qquad (\mu_S^2)_{\mathrm{I}} < (\lambda_R)_{\mathrm{I}}^2,$ as predicted in section 7.

(Here, of course, the eigenvalues are widely separated.)

A lemma

9. Hereafter we shall occasionally need to know the limits imposed on x by the double inequality

$$2 \leqslant x+x^{-1} \leqslant 2(1+\epsilon). \tag{19}$$

† The same equation governs the free vibrations of a tensioned string—a case considered in great arithmetical detail by Temple and Bickley ((**6**), section 5.2).

This requires that $x > 0$, so that $x^2+1 \leqslant 2(1+\epsilon)x$,—i.e.

$$\{x-(1+\epsilon)\}^2 \leqslant 2\epsilon+\epsilon^2;$$

therefore it implies that

$$x \text{ lies between } 1+\epsilon\pm\sqrt{(2\epsilon+\epsilon^2)}. \tag{20}$$

So also does the inequality

$$-2 \geqslant x+x^{-1} \geqslant 2(1+\epsilon), \tag{19)A}$$

which requires that $x < 0$; while the inequality

$$4 \leqslant (x+x^{-1})^2 \leqslant 4(1+\epsilon)^2, \tag{21}$$

which permits x to have either sign and requires that

$$2 \leqslant |x|+|x|^{-1} \leqslant 2(1+\epsilon), \text{ the \textit{positive} root of } 4(1+\epsilon)^2,$$

implies that $\quad |x| \text{ lies between } 1+\epsilon\pm\sqrt{(2\epsilon+\epsilon^2)}. \qquad (22)$

'Optimal synthesis' of w and $w_{\rm I}$

10. For a mode defined by

$$w_\gamma \equiv w+\gamma w_{\rm I}$$

(γ denoting an arbitrary constant) the 'Rayleigh estimate' is

$$(\lambda_R)_\gamma, \text{ say} \equiv \iint w_\gamma\, \vartheta w_\gamma\, dxdy \Big/ \iint w_\gamma\, \vartheta' w_\gamma\, dxdy$$

$$= \Lambda \iint w_\gamma\, \vartheta w_\gamma\, dxdy \Big/ \iint w_\gamma\, \vartheta(w_{\rm I}+\gamma w_{\rm II})\, dxdy$$

$$= \Lambda(I+2\gamma I'+\gamma^2 I'')/(I'+2\gamma I''+\gamma^2 I'''),$$

and the alternative estimate μ_S^2 is (similarly)

$$(\mu_S^2)_\gamma, \text{ say} = \Lambda^2(I+2\gamma I'+\gamma^2 I'')/(I''+2\gamma I'''+\gamma^2 I^{\rm iv}). \tag{23}$$

Therefore, as they depend on γ, $(\lambda_R)_\gamma$ has a stationary value when

$$I.I''-I'^2+\gamma(I.I'''-I'.I'')+\gamma^2(I'.I'''-I''^2) = 0, \tag{24}$$

$(\mu_S^2)_\gamma$ has a stationary value when

$$I.I'''-I'.I''+\gamma(I.I^{\rm iv}-I''^2)+\gamma^2(I'.I^{\rm iv}-I''.I''') = 0. \tag{25}$$

The lower stationary value of $(\lambda_R)_\gamma$ will normally (always, if the eigenvalues are all positive) be nearer λ_1 than either λ_R or $(\lambda_R)_{\rm I}$: and the lower stationary value of $(\mu_S^2)_\gamma$ will *always* be nearer λ_1^2 than either μ_S^2 or $(\mu_S^2)_{\rm I}$. When the curves representing (23) are flat (because w is a close approximation), their minima can be found by inserting trial values of γ close to the relevant root of (24) or (25).

An earlier paper (**8**) thus combined two guessed modes w_A and w_B (by 'optimal synthesis') to minimize λ_R and thereby to obtain a closer approximation to w_1 than either w_A or w_B. (In effect this is the 'device of the variable parameter'—apparently due to Rayleigh, though sometimes attributed to Ritz. (Cf. (**9**), section 526 and p. 500.) The γ's derived

from equations like (24) are always real and yield two modes which are conjugate: for the modes $(w+\gamma_1 w_I)$ and $(w+\gamma_2 w_I)$ will be conjugate, by (4), if

$$I+(\gamma_1+\gamma_2)I'+\gamma_1\gamma_2 I''=0, \qquad I'+(\gamma_1+\gamma_2)I''+\gamma_1\gamma_2 I'''=0,$$

and both conditions are satisfied when γ_1, γ_2 are the roots of (24), because then $(I'.I'''-I''^2)(\gamma_1+\gamma_2, \gamma_1\gamma_2) = (I'.I''-I.I'''), (I.I''-I'^2)$. On the other hand, when γ_1, γ_2 are the roots of (25), then

$$(I'.I^{\text{iv}}-I''.I''')(\gamma_1+\gamma_2, \gamma_1\gamma_2) = (I''^2-I.I^{\text{iv}}), (I.I'''-I'.I''),$$

and the first *but not the second* of (4) is satisfied.

Imposition of double limits on λ_1

First restrictive inequality

11. When I, I', I'' have been computed in accordance with (15), a value can be found for ϵ as defined by

$$\left.\begin{aligned} 2(1+\epsilon) &\equiv (I''+I)/I' = \textstyle\sum (B_k^2)/\sum (B_k^2/\lambda_k'), \\ B_k^2 &\equiv A_k^2(\Lambda^2/\lambda_k^2+1) > 0, \\ \lambda_k' &\equiv \Lambda/\lambda_k+\lambda_k/\Lambda. \end{aligned}\right\} \tag{26}$$

by (15), where

This equation has the form of (7), section 3, B_k^2 and λ_k' replacing A_k^2 and λ_k in that expression for λ_R: consequently Vaisey's theorem (section 4) can be adduced to prove that $2(1+\epsilon)$ cannot lie between the smallest positive and smallest negative λ_k, neither of which (since every λ is real) can lie between ± 2.† That is to say,

either $1+\epsilon \geqslant 1$, and then the smallest positive $\lambda' \leqslant 2(1+\epsilon)$,

or $1+\epsilon \leqslant -1$, and then the smallest negative $\lambda' \geqslant 2(1+\epsilon)$.

Suppose, then, that I' and hence $(1+\epsilon)$ have been found positive, and let x stand for that (positive) value of λ_k/Λ which yields the least positive λ'. Then

$$2 \leqslant x+x^{-1} \leqslant 2(1+\epsilon), \tag{19 bis}$$

and the argument of section 9 goes to show that

$$x \text{ lies between } 1+\epsilon \pm \sqrt{(2\epsilon+\epsilon^2)}. \tag{20 bis}$$

The same conclusion holds when I' and $(1+\epsilon)$ are negative, x then denoting that (negative) value of λ_k/Λ which yields the least negative λ'.

A refinement of the first restrictive inequality

12. The nearer $(1+\epsilon)$ is to ± 1, the narrower are the limits imposed on x by (20); and when w is replaced by w_γ as defined in (23), the value of ϵ is given by

$$2(1+\epsilon) = \{I+I''+2\gamma(I'+I''')+\gamma^2(I''+I^{\text{iv}})\}/(I'+2\gamma I''+\gamma^2 I''') \tag{27}$$

† $\lambda_k'^2 = 4+(\Lambda/\lambda_k-\lambda_k/\Lambda)^2$.

instead of by $(I+I'')/I'$. From this equation, as in section 10, two values of γ can be found for which ϵ takes a stationary value: they are the roots of

$$I''(I+I'')-I'(I'+I''')+\gamma\{I'''(I+I'')-I'(I''+I^{\text{iv}})\}+ \\ +\gamma^2\{I'''(I'+I''')-I''(I''+I^{\text{iv}})\} = 0, \tag{28}$$

and inserted in (27) they yield two values for ϵ of which the smaller may be used to impose limits on x.

Second restrictive inequality

13. Alternatively, a value can be attached to

$$\left.\begin{aligned} &4(1+\epsilon)^2 \equiv (I+2I''+I^{\text{iv}})/I'' = \sum (C_k^2/\lambda_k'^2), \text{ by (15)},\\ \text{where}\quad &C_k^2 \equiv A_k^2(\Lambda^2/\lambda_k^2+1)^2 > 0,\\ \text{and (as before)}\quad &\lambda_k' \equiv (\Lambda/\lambda_k+\lambda_k/\Lambda). \end{aligned}\right\} \tag{29}$$

This equation has the form of (13), section 7, λ_k^2 being now replaced by $\lambda_k'^2$; so Vaisey's theorem is no longer needed, because Rayleigh's original argument goes to show that the smallest λ'^2 (which $\nless 4$: cf. section 11, footnote) $\leqslant 4(1+\epsilon)^2$. Consequently if x stands for that value of λ_k/Λ which yields the smallest value of λ'^2, then (section 9)

$$|x| \text{ lies between } 1+\epsilon\pm\sqrt{(2\epsilon+\epsilon^2)}, \tag{22) \textit{bis}$$

$(1+\epsilon)$ now denoting the *positive* root of $(1+\epsilon)^2$.

The limits thus imposed on x will be closer if, as will normally be the fact, ϵ is smaller as given by (29) than as given by (26). They could be made still narrower by minimizing $(I+I^{\text{iv}})/I''$ in the manner of section 12; but the resulting equation would involve I^{v} and I^{vi}, which are not known. In this respect (26) has an advantage over (29); and it has another in that it gives a definite sign to the λ in question.

A test to verify that the restricted eigenvalue is (or is very nearly equal to) λ_1

14. When x is shown to have a value close to 1, Λ must approximate closely to some positive eigenvalue: when x has a value close to -1, $-\Lambda$ must approximate to some negative eigenvalue. Thus in every case *some* eigenvalue can be 'bracketed' by double limits. By methods which a subsequent paper will discuss, it can be made almost certain that the gravest positive mode predominates in w, and hence that Λ approximates to λ_1: then $(1+\epsilon)$ may be expected to be positive, and in that event (20) will put double limits on λ_1.

Evidence in support of this expectation is obtainable. Let λ_s denote the eigenvalue to which Λ is closest, so that $x \equiv \lambda_s/\Lambda$ lies close to 1: then in the mode (w', say) defined by

$$w' \equiv w-xw_{\text{I}} = \sum [A_k\, w_k(1-\lambda_s/\lambda_k)], \text{ by (10)},$$

the component w_s is nearly eliminated, and the contribution of another mode λ_k (say) will be

also nearly eliminated if $\lambda_k \doteqdot \lambda_s$,

very little altered if $|\lambda_k/\lambda_s|$ is large, but

greatly increased if $|\lambda_k/\lambda_s|$ is small (and especially if $\lambda_k/\lambda_s < 0$).

If, then, any $|\lambda|$ exists which is appreciably less than $|\lambda_s|$, the corresponding mode will make a larger (relative) contribution to w' than to w, and as such will yield a smaller value for μ_S^2; consequently, if μ_S^2 has a greater value for w' than for w, then no $|\lambda|$ appreciably less than $|\lambda_s|$ can exist. (The relevant formula is the last of (23) with γ replaced by $-x$ as here defined.)

Exemplification by the uniform strut

15. In illustration of sections 11–14 we now apply their methods to the uniform strut (section 8), for which the I's are given *exactly* by (18). *We suppose that Λ has been identified with λ_R (= 10), so that $I' = I$ exactly.* On that understanding,

$$I = I' = 1/3, \qquad I'' = 170/504, \qquad I''' = 775/2268, \qquad I^{\text{iv}} = 17275/49896. \tag{30}$$

These values inserted in (26) give

$$\epsilon = 1/168 = 0{\cdot}005\,952\,381, \qquad \sqrt{(2\epsilon+\epsilon^2)} = 0{\cdot}109\,271\,189,$$

and so, according to (20), for some λ_k

$$x = \lambda_k/10 \text{ lies between } 1{\cdot}005\,952 \pm 0{\cdot}109\,271. \tag{i}$$

Inserted in (29) they give

$$(1+\epsilon)^2 = 1+247/67320 \doteqdot (1{\cdot}0018_3)^2,$$

and so, according to (22), for some λ_k

$$|x| = |\lambda_k|/10 \text{ lies between } 1{\cdot}0018_3 \pm 0{\cdot}0605_7. \tag{ii}$$

The limits (ii) are narrower than (i), but do not define the sign of the λ_k. On the other hand, (i) shows that some positive $\lambda_k/10$ has a value close to unity, so presumably (ii) defines a positive eigenvalue.† Moreover, the limits (i) can be narrowed in the manner of section 12.

Thus (27), when the values (30) are inserted, becomes

$$\epsilon = \tfrac{1}{168}(1+2\gamma/9+5\gamma^2/198)/(1+85\gamma/42+775\gamma^2/756),$$

† It is known, of course, in this instance that the eigenvalues are all positive; but that knowledge is not used here, because in this respect the problem is not representative.

giving ϵ a minimum value 0·000 091 565 898 when $\gamma = -10{\cdot}305$; and (20), when this minimum value is given to ϵ, requires that for some λ_k

$$x = \lambda_k/10 \text{ lies between } 1{\cdot}000\,09_2 \pm 0{\cdot}013\,53_3. \qquad \text{(iii)}$$

These limits are finer than either (i) or (ii), and the sign of λ_k is defined.

16. Now the test of section 14 can be applied: the criterion being μ_S^2 as given by

$$\mu_S^2 = \tfrac{1680}{17}(1-2x+85x^2/84)/(1-310x/153+3455x^2/3366)$$

with x restricted in accordance with (iii). This (since $x \doteqdot 1$) has a value greater than $(\mu_S^2)_w$ as found in section 8 (viz. 1680/17) in a ratio sensitive to x but not less than 13: hence it can be taken as almost certain that λ_1 is the restricted eigenvalue (for here we have other reasons for believing that λ_1 and λ_2 are not nearly equal).

On that understanding the limits can be made still closer, because (cf. section 7) any $\mu_S \geqslant \lambda_1$, and by 'optimal synthesis', in the manner of section 10, a minimum value 9·869 680 1 is obtained for $(\mu_S)_\gamma$ as given by the last of (23). Taking this as the upper limit, and combining it with the lower limit given by (iii), we have finally,

$$9{\cdot}86558 \leqslant \lambda_1 \leqslant 9{\cdot}86968 \qquad (31)$$

as the restriction imposed by our methods on λ_1 for the uniform strut.

By adoption of the mean value of the two limits in (31), the correct value (π^2) is reproduced with an error of about 0·02 per cent. *And the methods by which the limits were deduced nowhere require an exact solution of the problem.*

An example in which the eigenvalues have both signs

17. This paper had its origin in a recent attempt to improve the accuracy of a solution which L. Fox and J. R. Green obtained by relaxation methods in 1941 (reference **8**). A full description of the problem will be given elsewhere: it concerned the elastic stability of a flat-plate cantilever under loads which, since they induce both positive and compressive principal stress, could also entail instability if their sense were reversed. This means that the eigenvalues (critical load-intensities) have both signs,† so that Rayleigh's method (in its original form) is not applicable: moreover the gravest mode is characterized by nodal lines, so cannot be guessed with accuracy as it can in simpler problems. Fox and Green, by 'point relaxation' combined with 'optimal synthesis' (section 10), had obtained an estimate of λ_1 which they believed to be 'correct within about 1 per cent.'; but had attempted less accuracy in their determination of w_1, and had not

† This consequence was first pointed out by Dr. D. N. de G. Allen.

established that their accepted mode and eigenvalue are in fact the gravest.

Miss Vaisey, in the more recent investigation (not yet published) employed, along with methods specially devised, the procedure which has been here termed 'intensification'; but arrived repeatedly (on her finest 'net') at a point where closer approximation seemed impracticable unless further indications could be drawn from theory. This difficulty had not been resolved in June 1952. She had obtained a mode believed to approximate closely to w_1, and this mode (w) she had twice intensified to obtain w_{I} and w_{II}; but these, though they yielded confirmatory estimates of λ_1, both differed sensibly from the mode obtained by Fox and Green.

She had deduced from them the integrals defined in (15), identifying Λ with λ_R as deduced from w and so making I and I' exactly equal (as in section 15). Her other computed values gave

$$I''/I = 1{\cdot}0012_1, \qquad I'''/I = 1{\cdot}0019_3, \qquad I^{\mathrm{iv}}/I = 1{\cdot}0029_3 \tag{32}$$

(a 'dropped' figure such as 3 denoting some value between 2 and 4 but possibly nearer to 2 or to 4 than to 3).

18. These values inserted in (7) and (13) give

$$\begin{aligned}
\lambda_R/\Lambda \text{ as deduced from } w &\equiv I/I' = 1,\\
\lambda_R/\Lambda \quad ,, \quad ,, \quad w_{\mathrm{I}} &\equiv I''/I''' = 0{\cdot}999_3,\\
(\mu_S/\Lambda)^2 \quad ,, \quad ,, \quad w &\equiv I/I'' = (0{\cdot}999_4)^2,\\
(\mu_S/\Lambda)^2 \quad ,, \quad ,, \quad w_{\mathrm{I}} &\equiv I''/I^{\mathrm{iv}} = (0{\cdot}999_1)^2.
\end{aligned}$$

They thus confirm the prediction of section 6, that $\lambda_R > (\lambda_R)_{\mathrm{I}}$, also the predictions of section 7, that $\mu_S^2 > (\mu_S^2)_{\mathrm{I}}$ and that

$$\mu_S^2 < (\lambda_R)^2, \quad (\mu_S^2)_{\mathrm{I}} < (\lambda_R)^2_{\mathrm{I}}.$$

(This is partial verification of their accuracy.)

Inserted in (23) they give

$$(\lambda_R)_\gamma/\Lambda = \{(1+\gamma)^2+12{\cdot}_1\times 10^{-4}\gamma^2\}/\{(1+\gamma)^2+(24{\cdot}_2\gamma+19{\cdot}_3\gamma^2)\times 10^{-4}\}, \tag{i}$$

$$(\mu_S^2)_\gamma/\Lambda^2 = \{(1+\gamma)^2+12{\cdot}_1\times 10^{-4}\gamma^2\}/\{(1+\gamma)^2+ +(12{\cdot}_1+38{\cdot}_\epsilon\gamma+29{\cdot}_3\gamma^2)\times 10^{-4}\}, \tag{ii}$$

whence (24) becomes $\qquad 12{\cdot}_1+7{\cdot}_2\gamma-4{\cdot}9\gamma^2 = 0 \qquad$ (iii)

and (25) becomes $\qquad 7{\cdot}_2+5{\cdot}_1\gamma-2{\cdot}_1\gamma^2 = 0. \qquad$ (iv)

According to (i) and (iii),

$(\lambda_R)_\gamma/\Lambda$ has a maximum value $-2{\cdot}464\,150\,39$ when $\gamma = -0{\cdot}99914$ and a minimum value $0{\cdot}999\,140\,032_4$ when $\gamma = 2{\cdot}47$.

According to (ii) and (iv),

$(\mu_S^2)_\gamma/\Lambda^2$ has a maximum value $(2{\cdot}081\,307\,18)^2$ when $\gamma = -0{\cdot}99907$ and a minimum value $(0{\cdot}999\,118\,09)^2$ when $\gamma = 3{\cdot}40$. $\qquad$ (33)

Inserted in (26) the values (32) give

$$2(1+\epsilon) = 1+I''/I = 2{\cdot}0012_1:$$

therefore, according to (20),

$$x \text{ lies between } 1{\cdot}0006_0 \pm 0{\cdot}0347_1 \tag{34}$$

when x denotes that value of λ_k/Λ which yields the smallest positive $(x+x^{-1})$.

The alternative formula (29) of section 13 requires, when the values (32) are inserted, that

$$(1+\epsilon)^2 \doteqdot 4{\cdot}0053_5/4{\cdot}0048_4 \doteqdot 1{\cdot}000\,12_7:$$

therefore, according to (22),

$$|x| \text{ lies between } 1{\cdot}0000_6 \pm 0{\cdot}0112_7. \tag{35}$$

These are much closer limits; but they do not define the sign of x.

19. Before proceeding to the refinement of section 12 we apply the test described in section 14. Its criterion here is the value of μ_S^2 as given by

$$\mu_S^2/\Lambda^2 = \{(1-x)^2 \times 10^4 + 12{\cdot}_1 x^2\}/\{(1-x)^2 \times 10^4 + 12{\cdot}_1 - 38{\cdot}_6 x + 29{\cdot}_3 x^2\}$$

when the range of x is restricted by (34). This value $\nless 1{\cdot}568$; so it is λ_1, or some very nearly equal eigenvalue, to which (34) relates.

Now introducing the refinement, we insert the values (32) in (27) to obtain

$$2\epsilon = (12{\cdot}_1 - 9{\cdot}_8\gamma + 2{\cdot}_8\gamma^2)/\{(1+\gamma)^2 \times 10^4 + 24{\cdot}_2\gamma + 19{\cdot}_3\gamma^2\},$$

according to which

$2\epsilon \times 10^4$ takes a minimum value $0{\cdot}399\,023\,148\,8$ when $\gamma = 2{\cdot}208$

(and a maximum value $-50\,330{\cdot}1234$ when $\gamma = -0{\cdot}991$).

When the minimum value is inserted in (20), that inequality requires some value of λ_k/Λ ($= x$) to lie between the limits

$$1{\cdot}000\,019_{95} \pm 0{\cdot}006\,217. \tag{36}$$

If, then, the λ in question is λ_1, which cannot exceed the minimum $(\mu_S)_\gamma$ as given in (33), we have, finally,

$$0{\cdot}993\,80_3 \leqslant \lambda_1/\Lambda \leqslant 0{\cdot}999\,118, \tag{37}$$

the upper limit coming from (33), the lower from (36). The limits, now, are very close indeed: identification of λ_1 with their mean cannot entail an error greater than 0·27 per cent.

Miss Vaisey's value of Λ was 6·002, so the final limits derived from her computations are

$$5{\cdot}964\,805 \leqslant \lambda_1 \leqslant 5{\cdot}996\,706. \tag{38}$$

If the λ in question differs slightly from λ_1, it lies between

$$\Lambda(1{\cdot}000\,019_{95} \pm 0{\cdot}006\,217),$$

by (36); i.e. in the range

$$5{\cdot}964\,805 \leqslant \lambda \leqslant 6{\cdot}039\,434. \tag{39}$$

Fox and Green ((**8**), section 32) gave the figure 5·949; so this paper confirms their belief (cf. section 17) that their estimate was correct within 1 per cent. (According to (38) it is in error by 0·8 per cent. at most; and according to (39), if it is low by more than 1 per cent., then λ_1 and λ_2 are equal to within $\frac{1}{2}$ per cent. It may (and of course in theory should) be too high: in that event λ_1, λ_2 will have somewhat wider separation.)

20. When close limits have been thus imposed on λ_1, more can be learnt about the associated mode w_1 and about other modes which contribute sensibly to w. This aspect will be discussed in a subsequent paper.

In theory more could be learnt from three or more intensifications and from the resulting integrals of type (15); but in practice this further information will be less trustworthy since the cumulative errors will be larger. It would seem better, when the optimal approximation to w_1 has been found in the manner of section 10, to start afresh with this as the starting assumption.

Acknowledgements

21. Miss Vaisey's computations leading to (32) will be described more fully in a joint paper to follow. I am further indebted to her for written comments which have led me to explicit recognition (section 19) of the possibility that two (or more) of the gravest eigenvalues may be nearly equal. Further work done on the example of section 17 indicates that this is the fact as regards that problem: where, accordingly, λ_1 may be slightly less than 5·95 and λ_2 greater than 5·95 by less than 1·5 per cent. (There is independent evidence to suggest that λ_1 lies close to 5·94.) Grateful acknowledgement is due to the Clothworkers' Company for a grant to Imperial College which made Miss Vaisey's collaboration possible. For other computations which required a desk machine I am indebted to Miss M. Evenett and Miss D. Gould of that College. My thanks are also due to its Governing Body for permitting me after retirement to maintain my contact with its 'relaxation team'.

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MODES OF VIBRATION OF A SUSPENDED CHAIN†‡

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SUMMARY

In the following paper a method is described for calculating the characteristic frequencies of a suspended inextensible chain vibrating with small amplitude in the plane of the catenary forming the equilibrium configuration. An asymptotic solution of the linearized equations of motion is obtained such that the accuracy of the results increases as the mode number increases and/or, as the catenary becomes flatter.

1. Equations of motion

We consider an inextensible chain of length L suspended from two points at the same level at a distance a ($\leqslant L$) apart. We measure the arc length, s, to a given point on the chain from the mid-point of the chain, and denote

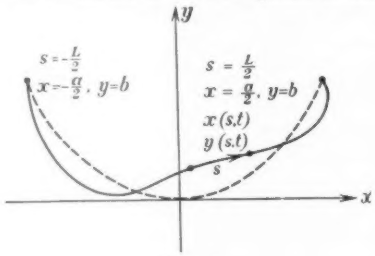

Fig. 1.

the space coordinates of this point by $x(s,t)$, $y(s,t)$, where x and y are referred to the lowest point of the equilibrium catenary as origin, as indicated in Fig. 1. We suppose that the end-points are located at $x=\pm a/2$; $y=b$. The equations of motion can now be written as follows (cf. Rohrs (**1**) and Pugsley (**2**)):

$$\frac{\partial^2 x}{\partial t^2}=\frac{\partial}{\partial s}\left(T\frac{\partial x}{\partial s}\right), \tag{1}$$

$$\frac{\partial^2 y}{\partial t^2}=\frac{\partial}{\partial s}\left(T\frac{\partial y}{\partial s}\right)-g. \tag{2}$$

Here $T(s,t)$ is the tension per unit mass per unit length in the string.

† The preparation of this paper was sponsored (in part) by the U.S. Office of Naval Research.

‡ The authors' interest in this hundred-year-old problem resulted from a conversation with Dr. E. U. Condon, then Director of the National Bureau of Standards. Dr. Condon had seen a reference to it in an article entitled 'Some experiments in dynamics, chiefly on vibrations' by John Satterly in the *American Journal of Physics*, **18** (1950), 405. Professor Satterly stated that he did not think that the calculation of the period of a swinging catenary swaying in its own plane had ever been done.

In addition to the equations of motion the chain is subject to an equation of constraint

$$\left(\frac{\partial x}{\partial s}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2 = 1. \tag{3}$$

The equations describing the equilibrium catenary are obtained by solving equations (1) to (3) with the time derivatives set equal to zero. These static equations are easily solved and, expressed for convenience in terms of the angle α which the tangent to the catenary at s makes with the x-axis, yield the following well-known equations

$$\left.\begin{aligned} x &= L\,\frac{\sinh^{-1}(\tan\alpha)}{2\tan\alpha_0} \\ y &= \frac{L}{2\tan\alpha_0}(\sec\alpha-1) \\ T &= \frac{gL}{2\tan\alpha_0}\,\frac{1}{\cos\alpha} \\ s &= \frac{L}{2}\,\frac{\tan\alpha}{\tan\alpha_0} \end{aligned}\right\}, \tag{4}$$

where α_0 is the angle at the end-point $(a/2, b)$.

2. Small amplitude approximation

We obtain a form of the equations of motion for the case of small amplitude vibrations by assuming a solution differing little from the equilibrium catenary. Thus we write

$$\left.\begin{aligned} x &= x_e(s)+\xi(s,t) \\ y &= y_e(s)+\eta(s,t) \\ T &= T_e(s)+\tau(s,t) \end{aligned}\right\}, \tag{5}$$

where the subscript e refers to the equilibrium values given in (4). Substituting expressions (5) into (1), (2), and (3) and neglecting quadratic terms in the small quantities ξ, η, τ, we obtain

$$\left.\begin{aligned} \frac{\partial^2\xi}{\partial t^2} &= \frac{\partial}{\partial s}\left(T_e\frac{\partial\xi}{\partial s}+\frac{dx_e}{ds}\tau\right) \\ \frac{\partial^2\eta}{\partial t^2} &= \frac{\partial}{\partial s}\left(T_e\frac{\partial\eta}{\partial s}+\frac{dy_e}{ds}\tau\right) \\ \frac{dx_e}{ds}\frac{\partial\xi}{\partial s}&+\frac{dy_e}{ds}\frac{\partial\eta}{\partial s} = 0 \end{aligned}\right\}. \tag{6}$$

For convenience, we change the independent variable to α by use of the last

of equations (4); we also assume ξ, η, τ are proportional to $\exp(-i\omega t)$. In this case equations (6) become:

$$-\lambda^2\xi = \cos^2\alpha \frac{d}{d\alpha}\left[\cos\alpha \frac{d\xi}{d\alpha} + (\cos\alpha)\tau\right], \tag{7}$$

$$-\lambda^2\eta = \cos^2\alpha \frac{d}{d\alpha}\left[\cos\alpha \frac{d\eta}{d\alpha} + (\sin\alpha)\tau\right], \tag{8}$$

$$\cos\alpha \frac{d\xi}{d\alpha} + \sin\alpha \frac{d\eta}{d\alpha} = 0, \tag{9}$$

where

$$\lambda^2 = \frac{L\omega^2}{2g\tan\alpha_0}. \tag{10}$$

This system of ordinary differential equations contains the three unknown functions ξ, η, τ and is to be solved subject to the boundary conditions $\xi, \eta = 0$, $\alpha = \pm\alpha_0$.

As an indication that the problem is completely defined, it can be shown that elimination of τ and η from (7), (8), and (9) leads to a third-order equation in ξ, the solution of which contains three arbitrary constants (see Appendix I). Determination of η, once ξ is known, is obtained by integrating (9), which adds a fourth constant to the solution. These four constants are just sufficient to satisfy the four boundary conditions, provided that λ satisfies the appropriate secular equation.

An important fact should be pointed out, namely,

$$\begin{aligned} \eta &= \text{constant} = C, \\ \xi &= 0, \end{aligned} \tag{11}$$

with

$$\tau = \frac{-\lambda^2 C}{\cos\alpha},$$

is an exact solution of equations (7), (8), and (9). This may be verified by substitution. Note, however, that $\eta = 0$, $\xi = \text{constant}$ is not a solution, although one might have been tempted to assume this to be so. This expresses the fact, which is not immediately obvious, that a hanging chain can be moved a small distance vertically without disturbing it from the equilibrium configuration, but that it cannot in like manner be moved sideways. Use of solution (11) will be made later.

For a better understanding of the complicated equations with which we are dealing, it is instructive to examine the limit of flat equilibrium catenaries, i.e. when $|\alpha| \leqslant \alpha_0 \ll 1$. In this limit, it can be shown that the solution is approximately (see Appendix II)

$$\begin{aligned} \eta &\simeq \frac{\sin}{\cos}(\lambda\alpha), \\ \xi &\simeq 0. \end{aligned} \tag{12}$$

The requirement that η vanish for $\alpha = \pm\alpha_0$ then gives

$$\lambda \simeq \begin{pmatrix} n \\ n+\frac{1}{2} \end{pmatrix} \frac{\pi}{\alpha_0}. \tag{13}$$

Although this result is only a first approximation, the essential features turn out to be correct. The major error is that actually no solution exists corresponding to $\lambda = \pi/2\alpha_0$ and the true lowest mode corresponds to

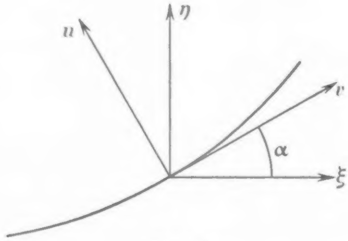

FIG. 2.

$\lambda = \pi/\alpha_0$. Otherwise these rough results turn out to be even quantitatively acceptable for $\alpha_0 \ll 1$ (see equations (33) and (34)).

Other important results follow investigation of this special case. From equation (13) it follows that, for small values of α_0, $\lambda \gg 1$. Further, since λ is roughly proportional to the mode number, for high enough mode numbers, $\lambda \gg 1$ for any α_0. Also, for small α_0, the waves are essentially transverse since the horizontal component of displacement ξ is small compared with the vertical component η. However, equation (9) suggests that the transversal character is true as long as λ is large. For high modes of a deep catenary, for example, $\xi \ll \eta$ near the bottom where α is small, while on the steep sides the opposite is true.

These characteristics make it seem reasonable to seek asymptotic solutions for $\lambda \gg 1$ since such solutions will be applicable over at least some interesting range of the configurations and mode numbers. Likewise it seems desirable to take advantage of the nearly transverse character of the waves by introducing new variables u and v, respectively, locally perpendicular and tangential to the equilibrium catenary, as indicated in Fig. 2. Thus, we substitute

$$\begin{aligned} u &= -\xi \sin\alpha + \eta\cos\alpha, \\ v &= \xi\cos\alpha + \eta\sin\alpha, \end{aligned} \tag{14}$$

into equations (7), (8), and (9) to obtain:

$$\frac{\lambda^2}{\cos^2\alpha}v = v\cos\alpha + u'\cos\alpha - \tau', \tag{15}$$

$$\frac{\lambda^2}{\cos^2\alpha}u = -u''\cos\alpha + u'\sin\alpha - u\cos\alpha - \tau + v\sin\alpha, \tag{16}$$

$$v' = u. \tag{17}$$

The primes indicate differentiation with respect to α. Elimination of u and τ from this system yields

$$(\cos\alpha)v^{(\mathrm{iv})} - (2\sin\alpha)v''' + \left(\cos\alpha + \frac{\lambda^2}{\cos^2\alpha}\right)v'' + \\ + \left(\frac{\lambda^2}{\cos^3\alpha} - 1\right)(2\sin\alpha)v' - \frac{\lambda^2}{\cos^2\alpha}v = 0. \tag{18}$$

Equation (18) is to be solved subject to the boundary conditions: $v = v' = 0$, when $\alpha = \pm\alpha_0$.

3. Asymptotic solutions

We next seek the four independent solutions of (18), at least asymptotically for large λ. The procedure which will be used is roughly equivalent to separating the fourth-order equation into two second-order equations, one of which leads to a pair of functions varying rapidly compared to the slowly varying trigonometric coefficients, the other leading to a pair of solutions which themselves vary slowly. That such a scheme is a suitable one can be seen quite directly from the following physical considerations. For large λ (or high mode numbers) we expect the problem to reduce substantially to one of transverse waves on a string with slowly varying tension. The asymptotic (WKB) solution for such a problem can be written down almost by inspection, of course, since it involves only a second-order differential equation for the transverse amplitude. Consequently, for large λ, we expect that it is possible to separate out of the rigorous equations which govern the motion a second-order equation leading to such a solution as a first approximation, and this indeed turns out to be the case. Use of these rapidly varying solutions iteratively furnishes higher order corrections which essentially take into account such factors as the curvature in the chain and the dynamic changes in the tension.

In addition to these corrections, and more important, the rigorous differential equation must, of course, have two other solutions of a different character which physically have their origin in the fact that the waves are not exactly transverse and the small, but not negligible, tangential displacements must also satisfy the boundary conditions. These then are the two

slowly varying solutions which are rather like two integration 'constants' required to fit the two additional boundary conditions. These slowly varying solutions cannot be found exactly, but they can be obtained asymptotically to any required degree of accuracy using iterative methods as we shall see.

We now follow the procedure indicated above, considering first the slowly varying solutions since they are simpler to treat. Because these functions presumably vary only trigonometrically with α, their derivatives are of the same order in λ as the functions themselves, and hence equation (18) is seen by inspection to consist simply of terms of order λ^2 and terms of order unity. As a first approximation, we retain only the terms of order λ^2 so that the differential equation reduces to the second-order equation

$$v''+(2\tan\alpha)v'-v = 0. \tag{19}$$

However, we already know one exact solution of (18), namely

$$v = C\sin\alpha, \qquad u = C\cos\alpha \tag{20}$$

corresponding to the solution $\xi = 0$, $\eta = C$ [see (11)]. This is, of course, a slowly varying solution, independent of λ; and hence satisfies (19), as well as (18), exactly. The construction of the second solution of (19) is thus trivial and we find at once

$$v = D(\cos\alpha+\alpha\sin\alpha), \qquad u = D\alpha\cos\alpha. \tag{21}$$

This then is the first approximation to the other slowly varying solution. To improve this approximation, one can establish a simple iterative procedure as follows. Denote by v_0 the solution (21) and expand v in the series

$$v = \sum v_n \lambda^{-2n}.$$

Substituting this series into (18) and equating coefficients of like powers of λ to zero, we obtain the system of equations

$$v_n''+2(\tan\alpha)v_n'-v_n$$
$$= -\cos^3\alpha\{v_{n-1}^{\mathrm{iv}}-2(\tan\alpha)v_{n-1}'''+v_{n-1}''-2(\tan\alpha)v_{n-1}'\},$$

so that each v_n can be determined in order from the preceding one. This procedure is easily carried out in principle since both solutions of the homogeneous equation, and hence the Green's function, are known. Thus, for example, we find in this way,

$$v_1 = -2D\left(1+\frac{\sin^4\alpha}{3}\right).$$

Actually, since the slowly varying solutions are essentially correction terms, we shall not require more than the first approximation (21).

Next we obtain asymptotic solutions for the two rapidly varying functions. In order to indicate the nature of the results we first show how to do this in a very rough way directly from equation (18). Since the rapidly varying functions are expected to be of the form $\exp[\pm i\lambda f(\alpha)]$, the nth derivative of such a function is to be regarded as of order λ^n times the function. Equation (18) then contains terms of order λ^4 through λ and retaining only the fourth-order terms, it reduces to

$$v^{(\mathrm{iv})}+\frac{\lambda^2}{\cos^3\alpha}v''=0,$$

which yields at once,

$$v\sim\exp\left(\pm i\lambda\int\frac{d\alpha}{\cos^{\frac{3}{2}}\alpha}\right)$$

and thus establishes the self-consistency of the assumptions.

Using this kind of argument, we could then analyse (18) more carefully to get more accurate results. Actually, however, it seems to be easier to accomplish this using equations (15), (16), and (17). The main reason is simply that, as seen from (17),

$$v\simeq O\left(\frac{1}{\lambda}u\right),$$

and thus v is not required as precisely as u. This can be regarded as an explicit statement of the fact that the waves are nearly transverse. Using the above relation, we see from (15) that

$$\tau\simeq O(u)$$

and hence, dropping terms of order u from (16), we obtain as the first approximate equation for u

$$u''-u'\tan\alpha+\frac{\lambda^2}{\cos^3\alpha}u=0 \tag{22}$$

and, as expected, this is precisely the equation for transverse vibrations on a string with tension T proportional to $1/\cos\alpha$. To order $1/\lambda$, the solution of (22) is easily found to be

$$u\simeq\cos^{\frac{1}{4}}\alpha\exp\left(\pm i\lambda\int_0^\alpha\frac{d\alpha}{\cos^{\frac{3}{2}}\alpha}\right). \tag{23}$$

We now use this result iteratively to obtain a solution to order $1/\lambda^2$. We can immediately get v to the required order using (17); thus

$$v\simeq\frac{\cos^{\frac{3}{2}}\alpha}{\pm i\lambda}u+O\left(\frac{1}{\lambda^2}u\right). \tag{24}$$

Using (15), we can now express τ in terms of u and after a little manipulation we find

$$\tau\simeq 2(\cos\alpha)u+O\left(\frac{1}{\lambda}u\right).$$

Substitution of these relations for τ and v into (16) then gives

$$u'' - (\tan\alpha)u' + \left[\frac{\lambda^2}{\cos^3\alpha} + 3 + O\left(\frac{1}{\lambda}\right)\right] u = 0.$$

Writing

$$u = \exp\left(\pm i\lambda \int_0^\alpha f_0(\alpha)\,d\alpha - \int_0^\alpha f_1(\alpha)\,d\alpha \pm i/\lambda \int_0^\alpha f_2(\alpha)\,d\alpha + \ldots\right)$$

we easily find the adiabatic solutions

$$u = \cos^{\frac{1}{4}}\alpha\, e^{\pm i(\lambda f + g/\lambda)} + O\left(\frac{1}{\lambda^2} u\right), \tag{25}$$

and from (24)

$$v = \frac{\cos^{\frac{7}{4}}\alpha}{\pm i\lambda} e^{\pm i(\lambda f + g/\lambda)} + O\left(\frac{1}{\lambda^2} u\right), \tag{26}$$

where

$$f(\alpha) = \int_0^\alpha \frac{d\alpha}{\cos^{\frac{3}{2}}\alpha} \tag{27}$$

and

$$g(\alpha) = \tfrac{11}{8} \int_0^\alpha (1 + \tfrac{1}{44}\tan^2\alpha)\cos^{\frac{3}{2}}\alpha\, d\alpha. \tag{28}$$

Using (20), (21), (25), and (26), we can then write the complete solutions, to order $1/\lambda^2$, as

$$u = A\cos^{\frac{1}{4}}\alpha\, e^{i(\lambda f + g/\lambda)} + B\cos^{\frac{1}{4}}\alpha\, e^{-i(\lambda f + g/\lambda)} + C\cos\alpha + D\alpha\cos\alpha,$$

$$v = \frac{A}{i\lambda}\cos^{\frac{7}{4}}\alpha\, e^{i(\lambda f + g/\lambda)} - \frac{B}{i\lambda}\cos^{\frac{7}{4}}\alpha\, e^{-i(\lambda f + g/\lambda)} + C\sin\alpha + D(\cos\alpha + \alpha\sin\alpha).$$

Under the requirement that u and v vanish at $\alpha = \pm\alpha_0$, the solutions decompose into odd and even pairs which lead immediately to the following transcendental equations for λ:

odd modes (u odd about $\alpha = 0$, v even)

$$\tan(\lambda f + g/\lambda) = -\frac{1}{\lambda}\frac{\alpha\cos^{\frac{5}{2}}\alpha}{\cos\alpha + \alpha\sin\alpha} \equiv -\frac{1}{\lambda}h(\alpha); \tag{29}$$

even modes (u even about $\alpha = 0$, v odd)

$$\cot(\lambda f + g/\lambda) = \frac{1}{\lambda}\frac{\cos^{\frac{5}{2}}\alpha}{\sin\alpha} \equiv \frac{1}{\lambda}k(\alpha). \tag{30}$$

For convenience, the subscript 0 has been dropped from α_0.

By observing that for $\lambda \gg 1$, we obtain $\lambda f \simeq n\pi$ and $\lambda f \simeq (n+\frac{1}{2})\pi$ respectively, we can easily obtain explicit solutions correct to order $1/\lambda^2$ by expanding. In this way we finally find

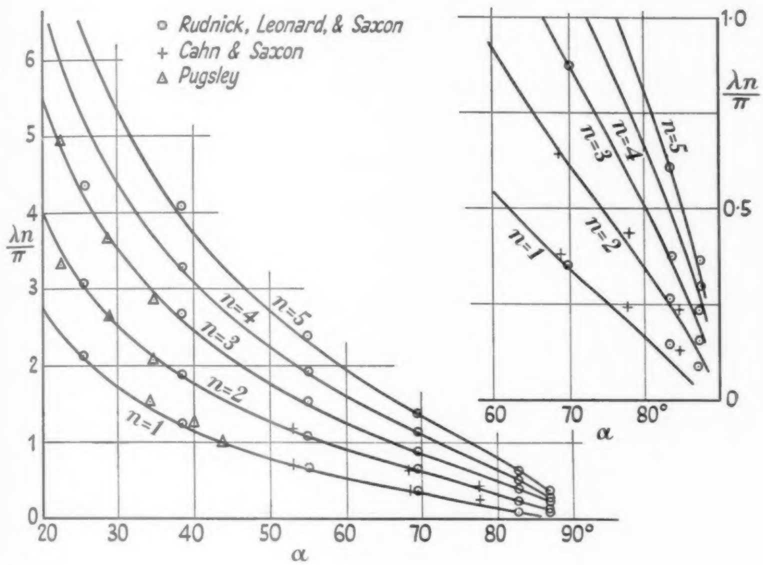

Fig. 3.

odd modes

$$\lambda \simeq \frac{n\pi}{f}\left[1-\frac{f(g+h)}{(n\pi)^2}+...\right] \quad (n = 1, 2,...); \tag{31}$$

even modes

$$\lambda \simeq \frac{(n+\frac{1}{2})\pi}{f}\left[1-\frac{f(g+k)}{[(n+\frac{1}{2})\pi]^2}+...\right] \quad (n = 1, 2,...). \tag{32}$$

We have thus obtained comparatively simple explicit expressions for λ, and hence, by (10), for the characteristic frequencies of the chain. The functions f, g, h, k are tabulated in Table I. The quantities λ_n/π for the first five modes, as determined from these functions, are tabulated in Table II and are plotted against α in Fig. 3. Included in this figure are the following experimental results:

(1) Data for the first five modes as determined with the kind assistance of Professors I. Rudnick and R. Leonard of the Department of Physics, University of California, Los Angeles. These points were

obtained by driving the chain at one end with a variable speed motor with an eccentric drive.

(2) Data for the first two modes vibrating freely as determined by the authors.

(3) Data for the first three modes as determined by Pugsley (**2**).

TABLE I

α	$f(\alpha) = \int_0^\alpha \frac{d\alpha}{\cos^{3/2}\alpha}$	$g(\alpha) = \frac{11}{8}\int_0^\alpha (1+\frac{1}{44}\tan^2\alpha)\cos^{3/2}\alpha\, d\alpha$	$h(\alpha) = \frac{\alpha\cos^{3/2}\alpha}{\cos\alpha+\alpha\sin\alpha}$	$k(\alpha) = \frac{\cos^{5/2}\alpha}{\sin\alpha}$
5°	·0874	·1198	·0861	11·3649
10°	·1759	·2382	·1655	5·5425
20°	·3601	·4660	·2821	2·5027
30°	·5630	·6738	·3240	1·3959
40°	·7994	·8534	·2952	·7990
50°	1·0949	·9992	·2205	·4324
60°	1·5015	1·1083	·1316	·2041
70°	2·1523	1·1814	·0561	·0728
75°	2·6883	1·2057	·0293	·0353
80°	3·5770	1·2235	·0113	·0128
82°	4·1455	1·2291	·0066	·0073
84°	4·9766	1·2343	·0033	·0036
86°	6·3681	1·2392	·0012	·0013
88°	9·5055	1·2446	·0002	·0002

TABLE II

α	$\frac{\lambda_1}{\pi}$	$\frac{\lambda_2}{\pi}$	$\frac{\lambda_3}{\pi}$	$\frac{\lambda_4}{\pi}$	$\frac{\lambda_5}{\pi}$
5°	11·4165	16·3802	22·8642	28·1279	34·3050
10°	5·6449	8·1383	11·3512	13·9803	17·0439
20°	2·7009	3·9645	5·5155	6·8215	8·3049
30°	1·6750	2·5244	3·5017	4·3565	5·2947
40°	1·1345	1·7647	2·4436	3·0602	3·7138
50°	·7897	1·2732	1·7648	2·2252	2·6987
60°	·5404	·9104	1·2692	1·6118	1·9562
70°	·3392	·6122	·8665	1·1107	1·3520
75°	·2469	·4742	·6814	·8797	1·0743
80°	·1545	·3358	·4966	·6488	·7970
82°	·1160	·2783	·4198	·5530	·6819
84°	·0755	·2178	·3392	·4522	·5610
86°	·0314	·1518	·2512	·3423	·4292
88°	·0209	·0737	·1473	·2126	·2736

As can be seen from the figure, agreement is quite satisfactory, even for the lowest mode for angles as large as 70°. As a final point, we observe that

for $\alpha \ll 1$, $f(\alpha) \simeq \alpha$ and since g, h are of order unity, for even modes we obtain

$$\lambda \simeq \frac{n\pi}{\alpha} \quad (\alpha \ll 1), \tag{33}$$

in agreement with the previous simple result. For odd modes, on the other hand, since $k(\alpha) \to 1/\alpha$ as $\alpha \to 0$, we obtain

$$\lambda = \frac{(n+\frac{1}{2})\pi}{\alpha}\left[1-\frac{1}{[(n+\frac{1}{2})\pi]^2}\right] \tag{34}$$

so that a small correction is required for these modes. It is interesting to note that $\lambda = n\pi/\alpha$, $\lambda = (n+\frac{1}{2})\pi/\alpha$ extrapolate exactly to the modes on a stretched string starting with the second mode. As might be expected, no analogue exists for the lowest string mode.

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APPENDIX I

We have

$$-\lambda^2\xi = \cos^2\alpha \frac{d}{d\alpha}[(\cos\alpha)\xi'+(\cos\alpha)\tau], \tag{7}$$

$$-\lambda^2\eta = \cos^2\alpha \frac{d}{d\alpha}[(\cos\alpha)\eta'+(\sin\alpha)\tau], \tag{8}$$

$$\xi'\cos\alpha+\eta'\sin\alpha = 0. \tag{9}$$

Expanding (7) and (8)

$$\frac{-\lambda^2\xi}{\cos^2\alpha} = \xi''\cos\alpha-\xi'\sin\alpha+\tau'\cos\alpha-\tau\sin\alpha, \tag{35}$$

$$\frac{-\lambda^2\eta}{\cos^2\alpha} = \eta''\cos\alpha-\eta'\sin\alpha+\tau'\sin\alpha+\tau\cos\alpha. \tag{36}$$

Eliminating τ' from (35) and (36)

$$\frac{-\lambda^2(\sin\alpha)\xi}{\cos^2\alpha}+\frac{\lambda^2\eta}{\cos\alpha} = (\sin\alpha\cos\alpha)\xi''-(\cos^2\alpha)\eta''-(\sin^2\alpha)\xi'+(\sin\alpha\cos\alpha)\eta'-\tau. \tag{37}$$

Eliminating η' and η'' by use of (9) and rearranging, (37) becomes

$$\tau\sin\alpha = (\cos\alpha)\xi''-\frac{1}{\sin\alpha}\xi'+\frac{\lambda^2\sin^2\alpha}{\cos^2\alpha}\xi-\frac{\lambda^2\sin\alpha}{\cos\alpha}\eta. \tag{38}$$

Substituting (38) into (8) and again using (9) we obtain

$$(\cos\alpha)\xi''' - \frac{2}{\sin\alpha}\xi'' + \left(\frac{2\cos\alpha}{\sin^2\alpha} + \cos\alpha + \frac{\lambda^2}{\cos^2\alpha}\right)\xi' + 2\lambda^2\frac{\sin\alpha}{\cos^3\alpha}\xi = 0, \tag{39}$$

which is the desired third-order equation in ξ.

APPENDIX II

For the case when $|\alpha| \leqslant \alpha_0 \ll 1$ we modify equations (7), (8), and (9)

$$-\lambda^2\xi = \cos^2\alpha\frac{d}{d\alpha}\left[\cos\alpha\frac{d\xi}{d\alpha} + \tau\cos\alpha\right], \tag{7}$$

$$-\lambda^2\eta = \cos^2\alpha\frac{d}{d\alpha}\left[\cos\alpha\frac{d\eta}{d\alpha} + \tau\sin\alpha\right], \tag{8}$$

$$\cos\alpha\frac{d\xi}{d\alpha} + \sin\alpha\frac{d\eta}{d\alpha} = 0 \tag{9}$$

by replacing $\sin\alpha$ by α and $\cos\alpha$ by 1 and thus obtain:

$$-\lambda^2\xi \simeq \frac{d^2\xi}{d\alpha^2} - \alpha\frac{d\xi}{d\alpha} + \frac{d\tau}{d\alpha} - \alpha\tau, \tag{40}$$

$$-\lambda^2\eta \simeq \frac{d^2\eta}{d\alpha^2} - \alpha\frac{d\eta}{d\alpha} + \alpha\frac{d\tau}{d\alpha} + \tau, \tag{41}$$

$$\frac{d\xi}{d\alpha} + \alpha\frac{d\eta}{d\alpha} \simeq 0. \tag{42}$$

In the limit of a flat equilibrium catenary, one might expect that the chain would behave similarly to a stretched string. If such is the case, then it is not hard to see that $\lambda = O(1/\alpha_0)$, and we here make that assumption. The approximate solution then obtained proves to be consistent with this.

In order to make evident the order of magnitude of the quantities involved, we introduce a new independent variable

$$w = \lambda\alpha \tag{43}$$

so that equations (40), (41), and (42) become

$$-\lambda^2\xi = \lambda^2\frac{d^2\xi}{dw^2} - w\frac{d\xi}{dw} + \lambda\frac{d\tau}{dw} - \frac{w}{\lambda}\tau, \tag{44}$$

$$-\lambda^2\eta = \lambda^2\frac{d^2\eta}{dw^2} - w\frac{d\eta}{dw} + w\frac{d\tau}{dw} + \tau, \tag{45}$$

$$\lambda\frac{d\xi}{dw} + w\frac{d\eta}{dw} = 0. \tag{46}$$

Since we are assuming $\lambda \gg 1$, let us consider solutions in the form of power series in λ^{-1}

$$\begin{aligned} \xi &= \sum_0 \xi_n\lambda^{-n}, \\ \eta &= \sum_0 \eta_n\lambda^{-n}, \\ \tau &= \sum_0 \tau_n\lambda^{-n}. \end{aligned} \tag{47}$$

Then equations (44), (45), and (46) yield sets of equations relating the terms of (47).

The first two sets of such equations are:

$$\left.\begin{aligned} -\xi_0 &= \frac{d^2\xi_0}{dw^2} \\ -\eta_0 &= \frac{d^2\eta_0}{dw^2} \\ \frac{d\xi_0}{dw} &= 0 \end{aligned}\right\} \tag{48}$$

$$\left.\begin{aligned} -\xi_1 &= \frac{d^2\xi_1}{dw^2}+\frac{d\tau_0}{dw} \\ -\eta_1 &= \frac{d^2\eta_1}{dw^2} \\ \frac{d\xi_1}{dw}+w\frac{d\eta_0}{dw} &= 0 \end{aligned}\right\}. \tag{49}$$

For the limit $1/\lambda \sim \alpha_0 \ll 1$, we are principally interested in the leading terms of (47), and from equations (48) and (49) these are seen to be

$$\left.\begin{aligned} \xi_0 &= 0 \\ \eta_0 &= \frac{\sin}{\cos}(w) = \frac{\sin}{\cos}(\lambda\alpha) \\ \tau_0 &= 2\frac{\sin}{\cos}(\lambda\alpha) = 2\eta_0 \end{aligned}\right\}. \quad (50)$$

Equations (50) give the results stated in equations (12) of the paper; equation (13) verifies the assumption that $\lambda = O(1/\alpha_0)$.

It might be mentioned that this procedure can be continued to obtain an improved approximation. However, in this case, it should be noted that more terms are required in the expansion of $\sin\alpha$ and $\cos\alpha$ in (7), (8), and (9).

VIRTUAL MOMENTUM AND SLENDER BODY THEORY

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SUMMARY

Munk's virtual momentum approach to the calculation of the transverse forces acting on slender pointed bodies is reconciled with the more rigorous analysis of Ward (as extended to unsteady flow) by comparing the dipole fields at infinity, using a method of solution due originally to Rayleigh. The virtual momentum of a given cross-section is exhibited in terms of the parameters associated with the mapping of this section on a circle.

1. Introduction

THE problem of the flow of an ideal fluid past a very slender body was first attacked by Munk (**1**), who used the concept of virtual momentum to determine the forces acting on airship hulls. More recently this same concept was applied by Jones (**2**) to very low aspect ratio pointed wings. These analyses were, in large measure, based on physical intuition, but a rigorous analysis of the supersonic steady flow problem has been given by Ward (**3**) and extended to unsteady flow by the writer (**4**).

It is not immediately obvious that Ward's results for a general cross-section are the same as those obtained on the basis of the virtual momentum concept.‡ The following development, which establishes the required identity, is based on Rayleigh's solution of a remarkably similar problem in diffraction (**5**). In the case of a body of revolution, we find an even earlier antecedent in Rayleigh's analysis of diffraction through a circular aperture (**6**).

2. Ward's formulation

We consider the supersonic flow of an ideal compressible fluid past a slender pointed body of length l located in the immediate neighbourhood of the s-axis, the undisturbed flow, U, being directed parallel to this axis. The coordinates (s, x, y) are a fixed Cartesian set, with s measured downstream. Then, consistent with the definition of a slender body (see ref. (**3**)

† This paper was written for the Aerodynamic Research group of Douglas Aircraft, Inc. (Santa Monica, Calif.), where the author is a consultant.

‡ Originally the writer was of the opinion that the (two-dimensional) virtual momentum approach was not generally valid. However, upon being informed by Mr. J. R. Spreiter that the approach did give Ward's result for a winged body, the present analysis suggested itself. It is understood that the matter is also being investigated by Mr. Alvin Sacks.

for details), the linearized equation for the velocity potential, ϕ, reduces to

$$\phi_{xx}+\phi_{yy} = 0. \tag{2.1}$$

All lengths, times, and velocities are dimensionless and referred to the characteristic quantities l, U, and l/U, respectively.

Ward's solution, as generalized in ref. (**4**), is developed in terms of the complex variable z and the complex potential w, defined by

$$z = x+iy, \tag{2.2}$$

$$w(s,x,y,t) = \phi+i\psi \tag{2.3 a}$$

$$= a_0(s,t)\ln z+b_0(s,t)+\sum_{m=1}^{\infty} a_m(s,t)z^{-m}. \tag{2.3 b}$$

The terms $a_0 \ln z+b_0$ are due essentially to the axial flow, and it is found (**3**, **4**) that

$$a_0 = \frac{1}{2\pi}\frac{DS}{Dt}, \tag{2.4}$$

where D/Dt indicates substantial time differentiation, viz.

$$\frac{D(\,)}{Dt} = \frac{\partial(\,)}{\partial s}+\frac{\partial(\,)}{\partial t}, \tag{2.5}$$

and S (which may depend on t as well as s) is the cross-sectional (transverse to s) area of the body. The term b_0 does not enter the calculation of the transverse forces, while the remaining terms in (2.3) can be determined by the solution of (2.1) for the cross-flow past the cross-section S at any particular station.

The extension (**4**) of Ward's calculation (**3**) of the transverse force components (X, Y) leads to the result

$$\frac{\partial}{\partial s}F(s,t) = \frac{\partial}{\partial s}(X+iY) \tag{2.6 a}$$

$$= \rho_0 U^2 \frac{D}{Dt}\left[2\pi a_1+\frac{D}{Dt}(z_g S)\right], \tag{2.6 b}$$

where ρ_0 is the mass density in the undisturbed flow, z_g the centroid of S, and a_1 the coefficient of z^{-1} in (2.3).

3. Solution in body coordinates

We now seek to relate the expression (2.6) to the virtual momentum at any local cross-section associated with the passage of a cylinder of this cross-section with velocity components (u, v) through a fluid at rest, corresponding to the translation

$$\frac{Dz_g}{Dt} = -(u+iv) = -q. \tag{3.1}$$

(We emphasize that u, v represent the prescribed velocity of the body, not the fluid velocity.) In establishing this relation it is expedient to introduce the complex body coordinate

$$z' = z - z_g(s,t) \tag{3.2}$$

and rewrite the solution (2.3) in the form

$$w = a_0 \ln z' + b_0 + w', \tag{3.3}$$

$$w' = \sum_1^\infty a'_m z'^{-m}, \tag{3.4}$$

where w' is the complex potential due to the two-dimensional cross-flow. The coefficient a_0 is still given by (2.4), as may be shown by repeating the derivation thereof in the body coordinates.

To relate the coefficient a'_1 to the desired coefficient a_1, we expand $\ln z'$ in inverse powers of z, viz.

$$\ln z' = \ln[z(1-z_g z^{-1})] = \ln z - z_g z^{-1} + \dots. \tag{3.5}$$

Then, since the leading term in the expansion of z'^{-1} in inverse powers of z is simply z^{-1}, we have

$$a_1 = a'_1 - a_0 z_g. \tag{3.6}$$

Substituting (3.1), (3.6), and (2.4) in (2.6 b) yields the force in the form

$$\frac{\partial F}{\partial s} = \rho_0 U^2 \frac{D}{Dt}(2\pi a'_1 - Sq). \tag{3.7}$$

It remains to determine the coefficient a'_1, which is the strength of the dipole field associated with the cross-flow w'.

4. Virtual momentum from dipole field

Let the (dimensionless) coefficients of virtual mass (A, H, B) of the cross-section S be defined by the expression

$$T = \tfrac{1}{2}\rho_0 U^2 l^2 (Au^2 + 2Huv + Bv^2) \tag{4.1}$$

for the kinetic energy of the flow field associated with the motion of translation prescribed by (3.1). The corresponding virtual momentum (dimensionless with $\rho_0 U l^2$ as reference quantity) is given by

$$M = M_x + iM_y = (Au+Hv) + i(Hu+Bv). \tag{4.2}$$

In terms of these coefficients, Lamb (**7**), following Rayleigh's analysis of the analogous diffraction problem (**5**), has shown that the distant field due to the motion is given by

$$w' \sim \{[(A+S)u+Hv] + i[Hu+(B+S)v]\}(2\pi z')^{-1}. \tag{4.3}$$

Equating the coefficient of z' to the dipole strength and substituting M from (4.2), we obtain

$$2\pi a'_1 = M + Sq. \tag{4.4}$$

Finally we substitute a_1' in (3.7) to obtain the desired result

$$\frac{\partial F}{\partial s} = \rho_0 U^2 \frac{DM}{Dt}, \tag{4.5}$$

which exhibits the transverse force on any local section as the rate of change of its virtual momentum.

5. Mapping solution

We conclude by constructing an expression for the virtual momentum in terms of the mapping of S on a circle. Let

$$\zeta = f(z') \tag{5.1}$$

be the conformal transformation that maps S on $|\zeta| = c(s,t)$. In order to preserve the flow at infinity, we impose the restriction

$$\lim_{z'\to\infty} z'^{-1}f(z') = 1. \tag{5.2}$$

Then, using the well-known solution for the flow (of complex velocity q at infinity) past a circle of radius c, viz.

$$w' = \bar{q}\zeta + qc^2\zeta^{-1} \tag{5.3}$$

where the bar indicates the complex conjugate, we find the total cross-flow in the z'-plane by substituting ζ from (5.1) and subtracting out the flow at infinity, namely $\bar{q}z'$, whence

$$w' = \bar{q}[f(z')-z'] + qc^2/f(z'). \tag{5.4}$$

The coefficient of z'^{-1} in the expansion of $f(z')$ will be designated as $\mathrm{Res}(f)$, while, by virtue of (5.2), the coefficient of z'^{-1} in the expansion of f^{-1} is simply unity. Hence we obtain

$$a_1' = \bar{q}\,\mathrm{Res} f + qc^2. \tag{5.5}$$

Substituting this in (4.4) and solving for M, we obtain

$$M = (2\pi c^2 - S)q + 2\pi\bar{q}\,\mathrm{Res} f. \tag{5.6}$$

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THE STEADY FLOW OF A VISCOUS FLUID PAST AN ELLIPTIC CYLINDER AND A FLAT PLATE AT SMALL REYNOLDS NUMBERS

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SUMMARY

In Part I the exact solution of Oseen's equations of motion for the steady flow of an incompressible viscous fluid past an elliptic cylinder is obtained, in terms of elliptic coordinates. The solution gives as a limiting case the flow past a flat plate.

The drag experienced by the elliptic cylinder is discussed in Part II, with special reference to both the pressure drag and the frictional drag. It is thus found that, as long as the calculations are based upon Oseen's equations, the total drag can be analysed into pressure and frictional drags proportional to the axes of the cylinder respectively for any value of the Reynolds number. Further, expansion formulae correct to the fourth power of the Reynolds number are derived for the drag on the elliptic cylinder and on a flat plate, and numerical values of the drag coefficient are computed from these formulae.

Part III deals with the general expression for the stream function representing the flow past the elliptic cylinder. An approximate formula for the stream function for the flow in the neighbourhood of the flat plate is then obtained. The velocity profiles as well as the stream-lines for a flat plate edgewise to the stream are drawn, while for a flat plate at right angles to the stream, the stream-lines show that there exists a pair of standing eddies behind the plate.

Lastly, the pressure distribution on the surface of the elliptic cylinder is calculated in Part IV.

1. Introduction

As is well known, the steady flow of an incompressible viscous fluid past a solid obstacle can be dealt with successfully on the basis of Oseen's linearized equations of motion when the Reynolds number is small. In previous papers (**1**, **2**) we have investigated in detail the steady flow past a sphere and a circular cylinder by making use of the exact analytical solution of Oseen's equations, and it has thus been found that the actual flow patterns observed around these obstacles can be deduced theoretically from this solution.

Similar investigations on the steady flow past an elliptic cylinder or a flat plate based on Oseen's equations have been made by several writers, but they were mainly concerned with discussions on the drag experienced by the obstacle. For instance, an approximate expression for the drag on an elliptic cylinder or a flat plate was obtained by Harrison (**3**), by Bairstow, Cave, and Lang (**4**), and by Piercy and Winny (**5**). Davies (**6**) deduced an

approximate formula for the drag on a flat plate edgewise to the uniform stream, but, as will be shown in this paper, his results are erroneous. Further, Sidrak (7) has also recently derived an approximate formula for the drag on an elliptic cylinder and on a flat plate, both placed parallel to the stream. But his analysis is vitiated by several errors and his results are unreliable.

As far as we are aware, no other detailed theoretical discussion based on Oseen's equations seems to have been made in the case of the steady flow around an elliptic cylinder or a flat plate placed parallel or perpendicularly to the uniform stream. The main object of the present paper is to discuss this problem and in particular to obtain the drag experienced by, and the flow patterns around, the obstacle by making use of the exact solution of Oseen's equations. The pressure distribution on the surface of the obstacle is also computed.

Detailed analytical calculations are given only for the case of an elliptic cylinder placed parallel to the stream, since the analysis for the case of the cylinder placed perpendicularly to the stream is similar to this case. However, essential parts of the analysis for the latter case are given in the Appendix.

PART I. SOLUTION OF OSEEN'S EQUATIONS

2. The general solution

We consider a two-dimensional cylindrical obstacle fixed in a steadily running uniform stream of an incompressible viscous fluid of infinite extent. Let $O(x, y)$ be the rectangular coordinates having the origin O within the obstacle and the axis of x in the direction of the velocity U of the uniform stream. We denote by $U+u$, v the components of the fluid velocity at any point, so that u, v are the velocity of perturbation which become vanishingly small everywhere at a great distance from the obstacle. If squares of u, v are omitted, we get, from the Navier–Stokes equations of motion, the well-known linearized equations of Oseen:

$$U\frac{\partial}{\partial x}(u, v) = -\frac{1}{\rho}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)p+\nu\nabla^2(u, v), \tag{1}$$

where p is the pressure, ρ the density, ν the kinematic viscosity, and ∇^2 stands for $\partial^2/\partial x^2+\partial^2/\partial y^2$. The fluid being assumed to be incompressible, the equation of continuity is then

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0. \tag{2}$$

If we put $k = U/2\nu$, these equations are satisfied by

$$u = -\frac{\partial \phi}{\partial x}+\frac{1}{2k}\frac{\partial \chi}{\partial x}-\chi, \qquad v = -\frac{\partial \phi}{\partial y}+\frac{1}{2k}\frac{\partial \chi}{\partial y}, \tag{3}$$

and
$$p = \rho U \frac{\partial \phi}{\partial x}, \tag{4}$$

provided that
$$\nabla^2 \phi = 0, \tag{5}$$

and
$$\nabla^2 \chi - 2k \frac{\partial \chi}{\partial x} = 0. \tag{6}$$

Consider an elliptic cylinder whose major and minor axes are of length $2a$ and $2b$ respectively, and let this cylinder be placed parallel to the uniform stream with its centre at the origin of coordinates and with the major and minor axes along the axes of x and y respectively. Let ξ and η be elliptic coordinates defined by

$$x + iy = c \cosh(\xi + i\eta), \tag{7}$$

where $c = \sqrt{(a^2 - b^2)}$.

Since the flow past the cylinder is symmetrical about the axis of x and perturbations due to the presence of the cylinder must vanish at infinity, the appropriate solution of equation (5) in terms of ξ and η is given by

$$\phi = cU\alpha_0 \xi - cU \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n e^{-n\xi} \cos n\eta, \tag{8}$$

where the α_n's are constants of integration.

To solve equation (6) put $\chi = e^{kx}\Xi(\xi)\mathrm{H}(\eta)$; the equation becomes separable in the variables and we are led to the modified Mathieu equation for $\Xi(\xi)$ and the Mathieu equation for $\mathrm{H}(\eta)$, namely

$$\frac{d^2\Xi}{d\xi^2} - \left(\lambda + \frac{\kappa^2}{2} \cosh 2\xi\right) \Xi = 0, \tag{9}$$

$$\frac{d^2\mathrm{H}}{d\eta^2} + \left(\lambda + \frac{\kappa^2}{2} \cos 2\eta\right) \mathrm{H} = 0, \tag{10}$$

where κ stands for kc. For a discrete set of characteristic values of λ the solutions of equation (10), which are periodic and even in η, are expressed as

$$\left.\begin{aligned} \mathrm{ce}_{2m}(\eta) &= \sum_{r=0}^{\infty} (-1)^{m+r} A_{2r}^{(2m)} \cos 2r\eta \\ \mathrm{ce}_{2m+1}(\eta) &= \sum_{r=0}^{\infty} (-1)^{m+r} B_{2r+1}^{(2m+1)} \cos(2r+1)\eta \end{aligned}\right\}, \tag{11}$$

where the coefficients $A_r^{(m)}$ and $B_r^{(m)}$ are functions of κ. The solutions of the second kind of equation (9), which are associated with $\mathrm{ce}_m(\eta)$, can be

expressed in terms of the modified Bessel functions I_n and K_n in the forms:†

$$\left.\begin{aligned} \mathrm{FEK}_{2m}(\xi) &= \frac{1}{A_0^{(2m)}} \sum_{r=0}^{\infty} A_{2r}^{(2m)} K_r(\tfrac{1}{2}\kappa e^{\xi}) I_r(\tfrac{1}{2}\kappa e^{-\xi}) \\ \mathrm{FEK}_{2m+1}(\xi) &= \frac{1}{B_1^{(2m+1)}} \sum_{r=0}^{\infty} B_{2r+1}^{(2m+1)} \times \\ &\qquad \times \{K_{r+1}(\tfrac{1}{2}\kappa e^{\xi}) I_r(\tfrac{1}{2}\kappa e^{-\xi}) - K_r(\tfrac{1}{2}\kappa e^{\xi}) I_{r+1}(\tfrac{1}{2}\kappa e^{-\xi})\} \end{aligned}\right\}. \tag{12}$$

These series involving Bessel function products are known to be rapidly convergent for small values of κ.

Thus the appropriate general expression for χ is given by

$$\chi = U e^{\kappa \cosh\xi \cos\eta} \sum_{m=0}^{\infty} \beta_m \mathrm{FEK}_m(\xi) \mathrm{ce}_m(\eta), \tag{13}$$

where the β_m's are constants of integration.

The components u, v of the perturbation velocity and the pressure p at any point in the flow can be obtained by inserting the above general expression for ϕ and χ into (3) and (4) respectively.

3. Determination of the constants of integration α_n and β_m

The constants of integration α_n and β_m can be determined from the boundary conditions at the surface of the elliptic cylinder. To do this it is convenient to introduce the components of the perturbation velocity v_ξ and v_η normal to an ellipse $\xi =$ constant and a hyperbola $\eta =$ constant respectively, which are orthogonal. These components are given, in terms of ϕ and χ, by

$$\left.\begin{aligned} v_\xi &= -\frac{1}{ch}\frac{\partial\phi}{\partial\xi} + \frac{1}{2\kappa h}\frac{\partial\chi}{\partial\xi} - \frac{1}{h}\chi \sinh\xi\cos\eta \\ v_\eta &= -\frac{1}{ch}\frac{\partial\phi}{\partial\eta} + \frac{1}{2\kappa h}\frac{\partial\chi}{\partial\eta} + \frac{1}{h}\chi\cosh\xi\sin\eta \end{aligned}\right\}, \tag{14}$$

where $$h^2 = \frac{1}{c^2}\frac{\partial(x,y)}{\partial(\xi,\eta)} = \cosh^2\xi - \cos^2\eta.$$

† The notations used here for the Mathieu function and the modified Mathieu function are similar to those in N. W. McLachlan's *Theory and Application of Mathieu Functions* (Oxford, 1947), except that our $\mathrm{FEK}_m(\xi)$ differs from McLachlan's function $\mathrm{Fek}_m(\xi)$ by a constant multiplier.

Substituting the expressions (8) and (13) for ϕ and χ into (14) we have

$$\left.\begin{aligned} v_\xi &= -\frac{U}{h}\sum_{n=0}^{\infty}\alpha_n e^{-n\xi}\cos n\eta+\frac{U}{2\kappa h}e^{\kappa\cosh\xi\cos\eta}\sum_{m=0}^{\infty}\beta_m\times \\ &\qquad \times\{\mathrm{FEK}'_m(\xi)\mathrm{ce}_m(\eta)-\kappa\sinh\xi\,\mathrm{FEK}_m(\xi)\mathrm{ce}_m(\eta)\cos\eta\} \\ v_\eta &= -\frac{U}{h}\sum_{n=1}^{\infty}\alpha_n e^{-n\xi}\sin n\eta+\frac{U}{2\kappa h}e^{\kappa\cosh\xi\cos\eta}\sum_{m=0}^{\infty}\beta_m\times \\ &\qquad \times\{\mathrm{FEK}_m(\xi)\mathrm{ce}'_m(\eta)+\kappa\cosh\xi\,\mathrm{FEK}_m(\xi)\mathrm{ce}_m(\eta)\sin\eta\} \end{aligned}\right\}, \tag{15}$$

where dashes denote differentiation with respect to ξ or η.

We now rewrite these expressions in forms which are suitable for fitting the boundary conditions at the surface of the cylinder and for calculating the stream function. For this purpose we expand the right-hand sides in (15) into Fourier series. Let us firstly introduce functions $F_{m,n}(\xi)$, $G_{m,n}(\xi)$, such that

$$\left.\begin{aligned} &\frac{1}{2\kappa}e^{\kappa\cosh\xi\cos\eta}\{\mathrm{FEK}'_m(\xi)\mathrm{ce}_m(\eta)-\kappa\sinh\xi\,\mathrm{FEK}_m(\xi)\mathrm{ce}_m(\eta)\cos\eta\} \\ &\qquad\qquad = \sum_{n=0}^{\infty}F_{m,n}(\xi)\cos n\eta \\ &\frac{1}{2\kappa}e^{\kappa\cosh\xi\cos\eta}\{\mathrm{FEK}_m(\xi)\mathrm{ce}'_m(\eta)+\kappa\cosh\xi\,\mathrm{FEK}_m(\xi)\mathrm{ce}_m(\eta)\sin\eta\} \\ &\qquad\qquad = \sum_{n=1}^{\infty}G_{m,n}(\xi)\sin n\eta \end{aligned}\right\}. \tag{16}$$

Then $F_{m,0}$ is clearly given by

$$F_{m,0} = -\frac{1}{2\kappa}\{\mathrm{FEK}_m(\xi)\mathrm{Ce}'_m(\xi)-\mathrm{FEK}'_m(\xi)\mathrm{Ce}_m(\xi)\},$$

where
$$\mathrm{Ce}_m(\xi) = \frac{1}{\pi}\int_0^{\pi} e^{\kappa\cosh\xi\cos\eta}\,\mathrm{ce}_m(\eta)\,d\eta,$$

and represents the definite integral form of the solution of the first kind of the modified Mathieu equation (9) associated with $\mathrm{ce}_m(\eta)$. Therefore $F_{m,0}$ is a constant independent of ξ but dependent upon m. This constant value can easily be evaluated by assuming a large value for ξ and replacing $\mathrm{FEK}_m(\xi)$ and $\mathrm{Ce}_m(\xi)$ by their respective asymptotic expressions, thus:

$$\mathrm{FEK}_m(\xi) \sim \sqrt{\left(\frac{\pi}{2z}\right)}e^{-z}, \qquad \mathrm{Ce}_m(\xi) \sim \mathrm{ce}_m(0)\frac{1}{\sqrt{(2\pi z)}}e^{z},$$

where $z = \frac{1}{2}\kappa e^{\xi}$. Hence we have

$$F_{m,0} = -\frac{1}{2\kappa}\mathrm{ce}_m(0). \tag{17}$$

Straightforward calculations give the expressions for $F_{m,n}(\xi)$ and $G_{m,n}(\xi)$ $(n = 1, 2, 3,...)$ as

$$\left.\begin{aligned} F_{m,n}(\xi) &= \frac{1}{\kappa}\{\mathrm{FEK}'_m(\xi)I_{m,n}(\xi)-\mathrm{FEK}_m(\xi)I'_{m,n}(\xi)\} \\ G_{m,n}(\xi) &= \frac{1}{\kappa}\mathrm{FEK}_m(\xi)\{\kappa\cosh\xi\, I_{m,n-1}(\xi)-nI_{m,n}(\xi)-\kappa\cosh\xi\, I_{m,n+1}(\xi)\} \end{aligned}\right\}, \tag{18}$$

where

$$\left.\begin{aligned} I_{2m,n}(\xi) &= \tfrac{1}{2}\sum_{r=0}^{\infty}(-1)^{m+r}A_{2r}^{(2m)}\{I_{n-2r}(\kappa\cosh\xi)+I_{n+2r}(\kappa\cosh\xi)\} \\ I_{2m+1,n}(\xi) &= \tfrac{1}{2}\sum_{r=0}^{\infty}(-1)^{m+r}B_{2r+1}^{(2m+1)}\{I_{n-2r-1}(\kappa\cosh\xi)+I_{n+2r+1}(\kappa\cosh\xi)\} \end{aligned}\right\}. \tag{19}$$

With these expressions for $F_{m,0}$, $F_{m,n}(\xi)$, and $G_{m,n}(\xi)$ $(n = 1, 2, 3,...)$ the expressions for v_ξ and v_η become

$$\left.\begin{aligned} v_\xi &= -\frac{U}{h}\sum_{n=0}^{\infty}\left\{\alpha_n e^{-n\xi}-\sum_{m=0}^{\infty}\beta_m F_{m,n}(\xi)\right\}\cos n\eta \\ v_\eta &= -\frac{U}{h}\sum_{n=1}^{\infty}\left\{\alpha_n e^{-n\xi}-\sum_{m=0}^{\infty}\beta_m G_{m,n}(\xi)\right\}\sin n\eta \end{aligned}\right\}. \tag{20}$$

The constants of integration α_n and β_m contained in (20) are determined by the boundary conditions. Since, however, the conditions at infinity are satisfied automatically, we have only to consider the conditions at the surface of the elliptic cylinder.

Let the equation of the surface of the elliptic cylinder under consideration be $\xi = \xi_0$, then the boundary conditions become

$$v_\xi = -\frac{U}{h}\sinh\xi\cos\eta, \qquad v_\eta = \frac{U}{h}\cosh\xi\sin\eta,$$

at $\xi = \xi_0$. Using the expressions (20) for v_ξ and v_η, these conditions yield immediately

$$\left.\begin{aligned} \alpha_0 &= -\frac{1}{2\kappa}\sum_{m=0}^{\infty}\mathrm{ce}_m(0)\,\beta_m \\ \alpha_n e^{-n\xi_0}-\sum_{m=0}^{\infty}\beta_m F_{m,n}(\xi_0) &= \begin{cases}\sinh\xi_0 & (n=1)\\ 0 & (n=2,3,...)\end{cases} \\ \alpha_n e^{-n\xi_0}-\sum_{m=0}^{\infty}\beta_m G_{m,n}(\xi_0) &= \begin{cases}-\cosh\xi_0 & (n=1)\\ 0 & (n=2,3,...)\end{cases} \end{aligned}\right\}. \tag{21}$$

Eliminating the α_n's from the second and third of these equations, we have

$$\sum_{m=0}^{\infty}\beta_m\{F_{m,n}(\xi_0)-G_{m,n}(\xi_0)\} = \begin{cases}-e^{\xi_0} & (n=1),\\ 0 & (n=2,3,...),\end{cases} \tag{22}$$

and the β_m's can be determined by solving this system of simultaneous linear algebraic equations.

PART II. THE DRAG ON THE ELLIPTIC CYLINDER AND ON THE FLAT PLATE

4. The general expression for the drag

We shall next calculate the drag experienced by an elliptic cylinder and also by a flat plate. As was mentioned in (1), two different methods are commonly used; either the drag is obtained by summing up the viscous stresses exerted by the fluid upon the surface of the obstacle, or it is calculated by applying the theorem of momentum to an infinite mass of fluid surrounding the obstacle.

If the analysis is based upon Oseen's equations, the first method gives the drag D on a cylindrical obstacle in the form

$$D = -\rho U \int \frac{\partial \phi}{\partial n}\, ds, \tag{23}$$

where integration is taken round the circumferential curve s of the cylinder and $\partial/\partial n$ denotes differentiation along the outward normal n to s. The second method gives

$$D = -\rho U \int \frac{\partial \phi}{\partial n'}\, d\sigma, \tag{24}$$

where integration is taken round a large closed contour σ everywhere at a great distance from the cylinder and $\partial/\partial n'$ denotes differentiation along the outward normal n' to σ.

For the elliptic cylinder, $\xi = \xi_0$, (23) becomes

$$D = -\rho U \int_0^{2\pi} \left(\frac{\partial \phi}{\partial \xi}\right)_{\xi=\xi_0} d\eta,$$

whilst (24) becomes $$D = -\rho U \int_0^{2\pi} \left(\frac{\partial \phi}{\partial \xi}\right)_{\xi\to\infty} d\eta,$$

σ being taken to be a large ellipse confocal with the cross-section of the cylinder.

Making use of the expression (8) for ϕ the above two formulae both give for the drag on the cylinder the value

$$D = -2\pi\rho U^2 c\alpha_0, \tag{25}$$

which, by the aid of the first of equations (21), can also be written in the form

$$D = 2\pi\mu U \sum_{m=0}^{\infty} \mathrm{ce}_m(0)\,\beta_m, \tag{26}$$

where μ denotes the coefficient of viscosity of the fluid. This gives the general expression for the drag on the elliptic cylinder placed parallel to the uniform stream.

Defining the drag coefficient C_D by $C_D = D/(\rho U^2 . 2a)^{-1}$, we have

$$C_D = \frac{2\pi}{R} \sum_{m=0}^{\infty} \mathrm{ce}_m(0)\,\beta_m, \tag{27}$$

where $R = 2aU/\nu$ is the Reynolds number.

The general expression for the drag on an elliptic cylinder placed perpendicularly to the uniform stream can be obtained likewise, as will be shown in the Appendix.

5. The pressure drag and the frictional drag

As was shown in (**1**), the total drag D on a solid can be analysed into the pressure drag D_p and the frictional drag D_f. In the case of a cylindrical body, adopting Oseen's approximation, the general formulae for D_p and D_f are given respectively by

$$D_p = -\rho U \int l \frac{\partial \phi}{\partial x}\, ds, \qquad D_f = -\rho U \int m \frac{\partial \phi}{\partial y}\, ds, \tag{28}$$

where (l, m) are the direction-cosines of the outward normal to the cross-sectional curve s of the cylinder.

In the particular case of an elliptic cylinder these become

$$D_p = -\rho U c \sinh \xi_0 \int_0^{2\pi} \left(\frac{\partial \phi}{\partial x}\right)_{\xi=\xi_0} \cos \eta \, d\eta,$$

$$D_f = -\rho U c \cosh \xi_0 \int_0^{2\pi} \left(\frac{\partial \phi}{\partial y}\right)_{\xi=\xi_0} \sin \eta \, d\eta,$$

and substituting the expression (8) for ϕ and carrying out integrations, we have

$$D_p = -2\pi\rho U^2 c \alpha_0 e^{-\xi_0} \sinh \xi_0,$$

$$D_f = -2\pi\rho U^2 c \alpha_0 e^{-\xi_0} \cosh \xi_0.$$

Adding these we immediately get again the expression (25) for the total drag D.

Further, remembering that $a = c\cosh \xi_0$ and $b = c \sinh \xi_0$ we arrive at an interesting result that

$$D_p = \frac{b}{a+b} D, \qquad D_f = \frac{a}{a+b} D. \tag{29}$$

This result shows that, if calculations are based upon Oseen's equations of motion, the total drag on the elliptic cylinder placed parallel to the uniform stream is divided into the pressure and frictional drags in the ratio $b : a$, whatever the value of the Reynolds number may be.

On the other hand, in a case when the elliptic cylinder is placed perpendicularly to the uniform stream we have

$$D_p = \frac{a}{a+b}D, \qquad D_f = \frac{b}{a+b}D. \tag{30}$$

Thus in this case the contribution of the pressure and frictional drags to the total drag are in the ratio $a:b$.

We can now deduce the results for the special cases of a flat plate and a circular cylinder. Thus, for a flat plate edgewise to the uniform stream we have, by putting $b = 0$ in (29),

$$D_p = 0, \qquad D_f = D, \tag{31}$$

while for a flat plate at right angles to the uniform stream we get, by putting $b = 0$ in (30),

$$D_p = D, \qquad D_f = 0. \tag{32}$$

Further, in the case of a circular cylinder ($a = b$) we have, from either of (29) and (30),

$$D_p = \tfrac{1}{2}D, \qquad D_f = \tfrac{1}{2}D. \tag{33}$$

Thus, as we already pointed out in (**1**), the total drag on the circular cylinder is equally divided into the pressure and frictional drags.

6. Expansion formulae for the drag

In order to calculate numerical values of the drag coefficient directly from the general expression (27) we must first determine the β_m's numerically by solving the system of simultaneous algebraic equations (22) for each given value of the Reynolds number; the work is thus rather cumbersome. It is therefore desirable to derive an expansion formula in powers of the Reynolds number which may be conveniently used for the purpose of computing the drag.

To this end, we have to express β_m as well as $\mathrm{ce}_m(0)$ in powers of κ, or in powers of the Reynolds number R. To do this, we must expand the quantities $F_{m,n}(\xi_0)-G_{m,n}(\xi_0)$ in (22) in powers of κ by using the series expansions for the coefficients $A_r^{(m)}$ and $B_r^{(m)}$ in $\mathrm{ce}_m(\eta)$ as well as the series expansions for the modified Bessel functions I_n and K_n which occur in $F_{m,n}(\xi)$ and $G_{m,n}(\xi)$. Assuming R to be small it is thus found that β_0, β_1, β_2,... are of order 1, R^2, R^4,..., respectively,† while it is easily seen that the quantity $\mathrm{ce}_m(0)$ for any number of m is of order unity.

Using the above estimated orders we have derived an expansion formula for the drag coefficient correct to the order of R^4. The expansions for β_m or $F_{m,n}(\xi_0)-G_{m,n}(\xi_0)$ necessary for our derivation need not be given here and only the final result will be stated.

† In order to obtain the expansions for β_0, β_1, and β_2 correct to the order of R^4, it is necessary to solve the first five equations ($n = 1$–5) in (22).

For convenience we introduce a parameter defined as

$$\sigma = \frac{1-(b/a)}{1+(b/a)},$$

which depends on the shape of the elliptic cylinder. Then the expansion for the drag on the elliptic cylinder whose major axis is parallel to the stream is given by

$$\begin{aligned} C_D = \frac{4\pi}{RS}\Big[1-\frac{R^2}{32(1+\sigma)^2S}\times \\ \times\{S^2-\tfrac{1}{2}(1+2\sigma-\sigma^2)S+\tfrac{1}{48}(15-4\sigma-18\sigma^2-12\sigma^3-\sigma^4)\}- \\ -\frac{R^4}{32^2(1+\sigma)^4S^2}\{S^4-\tfrac{1}{12}(4-3\sigma^2-2\sigma^3)S^3-\tfrac{1}{48}(31\sigma+10\sigma^2-6\sigma^3+10\sigma^4+\sigma^5)S^2+ \\ +\tfrac{1}{2880}(280+1185\sigma-987\sigma^2-2065\sigma^3-90\sigma^4+525\sigma^5+85\sigma^6+3\sigma^7)S- \\ -\tfrac{1}{2304}(225-120\sigma-524\sigma^2-216\sigma^3+390\sigma^4+440\sigma^5+180\sigma^6+24\sigma^7+\sigma^8)\}\Big], \end{aligned} \tag{34}$$

where
$$S = \tfrac{1}{2}(1+\sigma)-\gamma-\log\frac{R}{8(1+\sigma)},$$

and $\gamma = 0{\cdot}57721...$ is Euler's constant.

A similar expansion formula for the drag coefficient has also been obtained for the case when the elliptic cylinder is situated in such a manner that its major axis is perpendicular to the direction of the uniform stream. The result will be given here for comparison. Thus,

$$\begin{aligned} C_D = \frac{4\pi}{RS}\Big[1-\frac{R^2}{32(1+\sigma)^2S}\times \\ \times\{S^2-\tfrac{1}{2}(1-2\sigma-\sigma^2)S+\tfrac{1}{48}(15+4\sigma-18\sigma^2+12\sigma^3-\sigma^4)\}- \\ -\frac{R^4}{32^2(1+\sigma)^4S^2}\{S^4-\tfrac{1}{12}(4-3\sigma^2+2\sigma^3)S^3+\tfrac{1}{48}(31\sigma-10\sigma^2-6\sigma^3-10\sigma^4+\sigma^5)S^2+ \\ +\tfrac{1}{2880}(280-1185\sigma-987\sigma^2+2065\sigma^3-90\sigma^4-525\sigma^5+85\sigma^6-3\sigma^7)S- \\ -\tfrac{1}{2304}(225+120\sigma-524\sigma^2+216\sigma^3+390\sigma^4-440\sigma^5+180\sigma^6-24\sigma^7+\sigma^8)\}\Big], \end{aligned} \tag{35}$$

where
$$S = \tfrac{1}{2}(1-\sigma)-\gamma-\log\frac{R}{8(1+\sigma)}.$$

It may be noted that the first approximation $C_D = 4\pi/(RS)$, obtained by taking only the first term in the square bracket in (34) or (35), is identical with that given by Harrison (**3**) and also by Bairstow, Cave, and Lang (**4**). Apart from this, our formulae (34) and (35) for the drag on the elliptic cylinder of any thickness ratio appear to be new.

In a recent paper (**7**) Sidrak has obtained similar expansions, as far as terms of order of R^2, for the drag on an elliptic cylinder and for the skin friction of a flat plate, each placed parallel to the direction of the uniform flow. A careful examination, however, has revealed several errors† in his analysis, so that his work is unreliable.

Expansion formulae for the drag on a flat plate and a circular cylinder can be readily obtained as limiting forms of the above formulae (34) and (35). Thus, in the first place, when a flat plate of length L $(= 2a)$ is placed edgewise along the uniform stream, we get, by putting $\sigma = 1$ in (34), an expansion formula for the skin-friction coefficient of the plate correct as far as terms of order R^4, in the form

$$C_D = \frac{4\pi}{RS}\left[1-\frac{1}{S}(S^2-S-\tfrac{5}{12})\frac{R^2}{128}-\frac{1}{S^2}(S^4+\tfrac{1}{12}S^3-\tfrac{23}{24}S^2-\tfrac{133}{360}S-\tfrac{25}{144})\frac{R^4}{128^2}\right], \tag{36}$$

where
$$S = 1-\gamma-\log\frac{R}{16} = 3{\cdot}1954-\log R,$$

and the Reynolds number, R, is defined as $R = UL/\nu$.

This formula may be compared with the similar expansion formulae obtained by other writers. In 1933 Piercy and Winny (**5**), following Bairstow, Cave, and Lang's method of analysis (**4**), obtained a second approximation (correct as far as terms of order R^2) for the skin-friction coefficient of the flat plate. When expanded fully in powers of R their formula becomes, in our notation,

$$C_D = \frac{4\pi}{RS}\left[1-\frac{1}{S}(S^2-S-\tfrac{5}{12})\frac{R^2}{128}\right].$$

Thus it is seen that up to the order of R^2 our formula (36) is identical with Piercy and Winny's formula. One of us (S. T.) has extended Piercy and Winny's analysis to a third approximation, and it has been confirmed that our formula (36) is correct up to the term in R^4.

Davies (**6**) has also derived an expansion formula up to the order of R^2 for the skin friction of the plate placed edgewise to the uniform stream. But, as is noted by himself, no agreement is found between the expressions by himself and by Piercy and Winny. Recently Yosinobu and one of us (S. T.) have examined carefully Davies's analysis and found that his result is incorrect. It seems certain that this is due to bad convergence of the series-form of the modified Mathieu function used by him. Indeed, it has been found that if use is made of the function $\mathrm{FEK}_m(\xi)$, defined by (12), for the modified Mathieu function, we can derive, by pursuing Davies's

† One of the errors occurs at the beginning of his analysis, the expressions for the rectangular components of velocity of perturbation u and v being, in fact, erroneous.

analysis, the correct second approximation for the skin friction of the flat plate, which is identical with Piercy and Winny's formula.†

Next, if we put $\sigma = 1$ in (35), we get an expansion formula, correct as far as terms of order R^4, for the drag on the flat plate placed at right angles across the uniform stream. We find

$$C_D = \frac{4\pi}{RS}\left[1-\frac{1}{S}(S^2+S+\tfrac{1}{4})\frac{R^2}{128}-\frac{1}{S^2}(S^4-\tfrac{1}{4}S^3+\tfrac{1}{8}S^2-\tfrac{1}{8}S-\tfrac{1}{16})\frac{R^4}{128^2}\right], \tag{37}$$

with
$$S = -\gamma-\log\tfrac{1}{16}R = 2{\cdot}1954-\log R,$$

where, as before, the Reynolds number, R, is defined as $R = UL/\nu$. This formula does not seem to have been obtained before.

Lastly, an expansion formula for the drag on the circular cylinder of diameter $d = 2a$ can be obtained by putting $\sigma = 0$ in either of our preceding formulae (34) or (35). The result is

$$C_D = \frac{4\pi}{RS}\left[1-\frac{1}{S}(S^2-\tfrac{1}{2}S+\tfrac{5}{16})\frac{R^2}{32}-\frac{1}{S^2}(S^4-\tfrac{1}{3}S^3+\tfrac{7}{72}S-\tfrac{25}{256})\frac{R^4}{32^2}\right], \tag{38}$$

where
$$S = \tfrac{1}{2}-\gamma-\log\tfrac{1}{8}R = 2{\cdot}0022-\log R,$$

and $R = Ud/\nu$ is the Reynolds number. As should be expected, this formula is in complete agreement with that obtained by us previously (**8**).

7. Numerical values of the drag coefficient

By making use of our expansion formulae (34)–(38) we shall now calculate numerical values of the drag coefficient C_D for the elliptic cylinder (including the flat plate and the circular cylinder as limiting cases) at small Reynolds numbers. The range of values of the Reynolds number in which these formulae can legitimately be used was first estimated as for the case of the expansion formula for the drag on the circular cylinder of our previous paper (**8**), and it has been found that all our formulae can be used with sufficient accuracy for values of the Reynolds numbers less than about 4.

Numerical values of C_D thus calculated are given in Table I and are shown graphically in Fig. 1. It is of some interest to note, as will readily be seen from Table I or Fig. 1, that the value of the drag coefficient of the circular cylinder at any small Reynolds number is always larger than the corresponding value for the elliptic cylinder or the flat plate, each having the same length as the diameter of the circular cylinder. This '*rather curious*' result has been already pointed out by Bairstow, Cave, and Lang (**4**, p. 429) by using the first approximation $C_D = 4\pi/(RS)$.

† Our analysis will be given elsewhere.

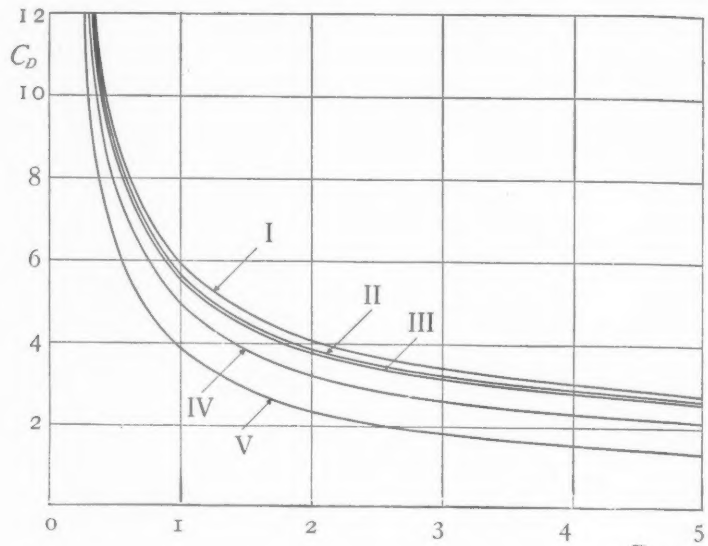

FIG. 1. Curves showing the variation of the total drag coefficient C_D with the Reynolds number R. Curves I, II, III, IV, and V correspond respectively to the cases D, E, G, C, and A in Table I.

TABLE I. *Values of* C_D

Case A. Flat plate
Case B. Elliptic cylinder ($b/a = \frac{1}{4}$)
Case C. Elliptic cylinder ($b/a = \frac{1}{2}$)
} parallel to the stream.

Case D. Circular cylinder.
Case E. Elliptic cylinder ($b/a = \frac{1}{2}$)
Case F. Elliptic cylinder ($b/a = \frac{1}{4}$)
Case G. Flat plate
} perpendicular to the stream.

R	*Case A*	*Case B*	*Case C*	*Case D*	*Case E*	*Case F*	*Case G*
0·4	7·61	8·47	9·24	10·63	10·25	10·10	10·04
0·6	5·61	6·31	6·95	8·13	7·82	7·70	7·66
0·8	4·54	5·06	5·73	6·78	6·51	6·41	6·38
1	3·87	4·43	4·95	5·93	5·69	5·60	5·57
2	2·39	2·85	3·24	4·07	3·88	3·84	3·83
3	1·84	2·28	2·62	3·47	3·22	3·20	3·18
4	1·55	2·04	2·35	2·92†	—	2·89	2·84

† This value 2·92 has been calculated by the general expression (given by (48) in our previous paper (**1**)) for the drag on the circular cylinder.

In the case of the flat plate placed edgewise to the uniform stream, the value of the drag coefficient for $R = 4$ has been computed by several writers

by using different methods. Thus Bairstow, Cave, and Lang (**4**) have obtained graphically the result $C_D = 1{\cdot}527$, while Southwell and Squire (**9**), using the extended Oseen's equations due to them, obtained numerically the value of $C_D = 1{\cdot}563$. These values may be compared with the value $C_D = 1{\cdot}551$ calculated from our formula (36). Although there appear to be slight discrepancies among these values, the agreement between them is nevertheless rather good, especially if we take account of probable errors which can arise in graphical computations. Lastly, it may be added here that Piercy and Winny (**5**) obtained the value $C_D = 1{\cdot}499$ for the skin-friction coefficient of the flat plate by using an asymptotic solution of Oseen's equations.

Part III. The Flow Patterns around the Elliptic Cylinder and the Flat Plate

8. The general expression for the stream function

We shall now calculate the general expression of the stream function for the flow past the elliptic cylinder by using the exact solution of Oseen's equations. If we introduce the stream function ψ' for the velocity of perturbation so that $u = \partial\psi'/\partial y$ and $v = -\partial\psi'/\partial x$, then v_ξ is expressed in terms of ψ' as $v_\xi = (ch)^{-1}\,\partial\psi'/\partial\eta$.

If we denote the stream function for the flow past the fixed cylinder in the uniform stream by ψ, we have

$$\psi = Uy + \int\limits_0^\eta chv_\xi\,d\eta,$$

provided that the constant of integration is so chosen that the axis of x forms a part of the stream-line $\psi = 0$.

Inserting in this equation the expression (20) for v_ξ, and taking account of the first and second of equations (21), we get

$$\psi = cU(\sinh\xi - e^{\xi_0-\xi}\sinh\xi_0)\sin\eta + \\ + cU\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\beta_m\{F_{m,n}(\xi) - e^{n(\xi_0-\xi)}F_{m,n}(\xi_0)\}\frac{\sin n\eta}{n}. \tag{39}$$

This is the required expression for the stream function for the flow past the elliptic cylinder whose major axis is parallel to the uniform stream. As will readily be seen, the first term on the right-hand side is the stream function in the case of a perfect fluid and consequently the second term represents the effect of viscosity of the fluid.

If we introduce the non-dimensional stream function ψ_1 defined as $\psi_1 = \psi/(Ua)$, we have

$$\psi_1 = \operatorname{sech}\xi_0(\sinh\xi - e^{\xi_0-\xi}\sinh\xi_0)\sin\eta +$$
$$+\operatorname{sech}\xi_0 \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\beta_m\{F_{m,n}(\xi) - e^{n(\xi_0-\xi)}F_{m,n}(\xi_0)\}\frac{\sin n\eta}{n}. \quad (40)$$

A similar general expression for the stream function in the case of flow past an elliptic cylinder placed at right angles to the uniform stream will be given in the Appendix.

9. Approximate formulae for the stream function in the case of flow around the flat plate

We shall now deduce from (40) an approximate formula for the stream function for small Reynolds numbers R, which will be used to compute the flow patterns.

To this end, the general expression for the stream function will be expanded in powers of R as far as terms of order R^2. We first derive expansions in powers of R of the functions $F_{m,n}(\xi)$ given by (40), and of the quantities β_m. Expansions of the β_m's have already been obtained. Expansions for the functions $F_{m,n}(\xi)$ have been obtained by using the series definitions of the modified Bessel functions I_n and K_n (which occur in the expressions for $F_{m,n}(\xi)$) on the assumption that their arguments are small. Accordingly such expansions are justified when the variable ξ is also small; our approximate formula for the stream function may be adequately used for the flow near the obstacle.

In this way, we have obtained an approximate formula for the stream function for the flow past the elliptic cylinder of any thickness ratio and we have deduced the corresponding formula for the case of a flat plate by making the parameter ξ_0 tend to zero. Since it is of special interest to discuss the flow patterns around the flat plate, only the results for this case will be given below.

For the flow past a flat plate placed edgewise to the uniform stream an approximate formula for the stream function, correct to the order R^2, is

$$\psi_1 = \{(a_{1,3}+b_{1,3}\xi)e^{3\xi}+(a_{1,1}+b_{1,1}\xi)e^{\xi}+(a_{1,-1}+b_{1,-1}\xi)e^{-\xi}+$$
$$+(a_{1,-3}+b_{1,-3}\xi)e^{-3\xi}\}\sin\eta+\{(a_{2,2}+b_{2,2}\xi)e^{2\xi}+a_{2,0}+(a_{2,-2}+b_{2,-2}\xi)e^{-2\xi}\}\sin 2\eta+$$
$$+\{(a_{3,3}+b_{3,3}\xi)e^{3\xi}+(a_{3,1}+b_{3,1}\xi)e^{\xi}+(a_{3,-1}+b_{3,-1}\xi)e^{-\xi}+$$
$$+(a_{3,-3}+b_{3,-3}\xi)e^{-3\xi}\}\sin 3\eta, \quad (41)$$

where the coefficients $a_{m,n}$ and $b_{m,n}$ are given by

$$\left.\begin{array}{ll}
a_{1,3} = -\dfrac{R^2}{1024}, & b_{1,3} = -b_{1,-3} = \dfrac{R^2}{1024}\dfrac{1}{S}, \\[2ex]
a_{1,1} = \dfrac{R^2}{512}\left(1-\dfrac{13}{12S}\right), & b_{1,1} = -b_{1,-1} = \dfrac{1}{2S}+\dfrac{R^2}{1024}\left(\dfrac{1}{S}+\dfrac{5}{3S^2}\right); \\[2ex]
a_{1,-1} = -\dfrac{R^2}{1024}\left(1-\dfrac{13}{3S}\right), & \\[2ex]
a_{1,-3} = -\dfrac{R^2}{512}\dfrac{13}{12S}, & \\[2ex]
a_{2,2} = a_{2,-2} = -\dfrac{R}{64}\left(1-\dfrac{1}{2S}\right), & b_{2,2} = -b_{2,-2} = \dfrac{R}{64}\dfrac{1}{S}; \\[2ex]
a_{2,0} = \dfrac{R}{32}\left(1-\dfrac{1}{2S}\right), & \\[2ex]
a_{3,3} = -\dfrac{R^2}{3072}\left(1-\dfrac{2}{3S}\right), & b_{3,3} = -b_{3,-3} = \dfrac{R^2}{3072}\dfrac{1}{S}, \\[2ex]
a_{3,1} = \dfrac{R^2}{2048}\dfrac{1}{S}, & b_{3,1} = -b_{3,-1} = \dfrac{R^2}{1024}\dfrac{1}{S}; \\[2ex]
a_{3,-1} = \dfrac{R^2}{1024}\left(1-\dfrac{5}{3S}\right), & \\[2ex]
a_{3,-3} = -\dfrac{R^2}{1536}\left(1-\dfrac{17}{12S}\right), &
\end{array}\right\} \tag{42}$$

the quantity S being defined in (36).

On the other hand, for the case of a flat plate placed at right angles to the uniform stream, the corresponding expansion formula, correct to the order of R^2, is of the same form as (41), but in this case the coefficients $a_{m,n}$ and $b_{m,n}$ are given by

$$\begin{array}{ll}
a_{1,3} = -\dfrac{R^2}{1024}\left(1+\dfrac{1}{S}\right), & b_{1,3} = b_{1,-3} = \dfrac{R^2}{1024}\dfrac{1}{S}, \\[2ex]
a_{1,1} = -\dfrac{1}{2S}+\dfrac{R^2}{256}\left(1+\dfrac{5}{8S}+\dfrac{1}{4S^2}\right), & b_{1,1} = b_{1,-1} = \dfrac{1}{2S}-\dfrac{R^2}{1024}\left(\dfrac{1}{S}+\dfrac{1}{S^2}\right); \\[2ex]
a_{1,-1} = \dfrac{1}{2S}-\dfrac{R^2}{1024}\left(5+\dfrac{2}{S}+\dfrac{1}{S^2}\right), & \\[2ex]
a_{1,-3} = \dfrac{R^2}{512}\left(1+\dfrac{1}{4S}\right), & \\[2ex]
a_{2,2} = a_{2,-2} = -\dfrac{R}{64}\left(1+\dfrac{1}{2S}\right), & b_{2,2} = -b_{2,-2} = \dfrac{R}{64}\dfrac{1}{S};
\end{array}$$

$$
\begin{aligned}
a_{2,0} &= \frac{R}{32}\left(1+\frac{1}{2S}\right), \\
a_{3,3} &= -\frac{R^2}{3072}\left(1+\frac{1}{3S}\right), \qquad & b_{3,3} = b_{3,-3} &= \frac{R^2}{3072}\frac{1}{S}, \\
a_{3,1} &= \frac{R^2}{512}\left(1+\frac{1}{4S}\right), \qquad & b_{3,1} = b_{3,-1} &= -\frac{R^2}{1024}\frac{1}{S}; \\
a_{3,-1} &= -\frac{3R^2}{1024}, \\
a_{3,-3} &= \frac{R^2}{768}\left(1-\frac{7}{24S}\right),
\end{aligned}
\tag{43}
$$

where the quantity S is given in (37).

10. The flow patterns around the flat plate

Using the above approximate formulae for the stream function, we shall now discuss the flow patterns around the flat plate. In the first place, we shall compute the velocity distributions along the lines perpendicular to the plate placed edgewise along the uniform stream, in a case when the Reynolds number is equal to 4. The results are given in Table II, where the values of the velocity components $v_x = U+u$ and $v_y = v$ along the lines $x = -1{\cdot}5,\ -1{\cdot}0,\ -0{\cdot}5,\ 0,\ 0{\cdot}5,\ 1{\cdot}0,\ 1{\cdot}5$ are given, taking, for convenience, the uniform velocity U to be unity and the length L of the plate to be 2. The velocity profiles are shown in Fig. 2, in which the stream-lines around the plate are also shown.

It will be seen immediately from this figure that when the Reynolds number is small, the flow near the flat plate resembles closely the shear flow with uniform velocity-gradient, except in the vicinity of its edges.

TABLE II. *The velocity distributions around the flat plate edgewise along the stream in the case when $R = 4$*

y	$x = -1{\cdot}5$ v_x	v_y	$x = -1{\cdot}0$ v_x	v_y	$x = -0{\cdot}5$ v_x	v_y	$x = 0$ v_x	v_y	$x = 0{\cdot}5$ v_x	v_y	$x = 1{\cdot}0$ v_x	v_y	$x = 1{\cdot}5$ v_x	v_y
0	0·54	0	0	0	0	0	0	0	0	0	0	0	0·53	0
0·1	0·6	0·0	0·29	0·09	0·13	0·01	0·10	0·00	0·11	−0·00	0·22	−0·07	0·5	−0·1
0·2	0·6	0·1	0·39	0·13	0·24	0·02	0·19	0·00	0·21	−0·01	0·33	−0·11	0·6	−0·1
0·3	—	—	0·47	0·15	0·34	0·04	0·29	0·01	0·30	−0·02	0·42	−0·13	—	—
0·4	—	—	0·53	0·16	0·43	0·06	0·37	0·01	0·39	−0·04	0·50	−0·16	—	—
0·5	—	—	0·60	0·18	0·50	0·08	0·45	0·01	0·47	−0·05	0·55	−0·17	—	—

The stream-lines around a flat plate placed at right angles to the uniform stream have been drawn in case when $R = 4$. The flow patterns thus

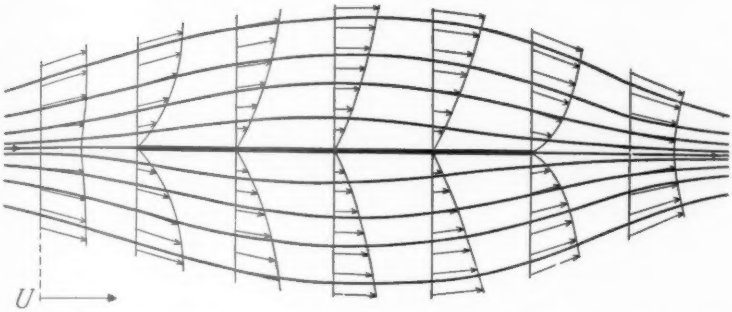

FIG. 2. Calculated stream-lines and velocity profiles for $R = 4$ around a flat plate placed edgewise along the uniform stream of velocity U. In the half-field of flow above the central stream-line $\psi_1 = 0$, the stream-lines correspond respectively to $\psi_1 = 0{\cdot}01$, $0{\cdot}05$, $0{\cdot}1$, and $0{\cdot}2$.

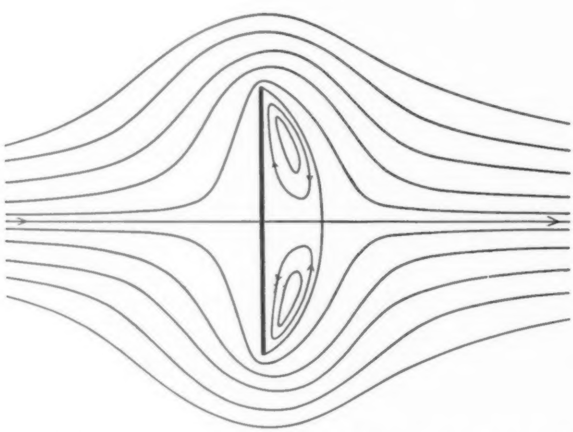

FIG. 3. Calculated stream-lines past a flat plate at right angles across the uniform stream for $R = 4$. In the half-field of flow above the central stream-line $\psi_1 = 0$, the stream-lines correspond respectively to $\psi_1 = 0{\cdot}01$, $0{\cdot}05$, $0{\cdot}1$, $0{\cdot}2$, and $0{\cdot}3$, while the stream-lines in the region of eddies are at $\psi_1 = -0{\cdot}0005$ and $-0{\cdot}001$ respectively.

obtained are shown in Fig. 3; it will be seen that there exists a pair of standing eddies behind the plate. It is worth noticing that when the flat plate is placed at right angles to the stream, a pair of standing eddies will always appear behind the body even when the Reynolds number is very small.

PART IV. THE PRESSURE DISTRIBUTIONS ON THE SURFACE OF THE ELLIPTIC CYLINDER

11. The pressure distributions

Lastly we shall compute the pressure distributions on the surface of the elliptic cylinder. For this purpose we first derive from the exact solution of Oseen's equations the general expression for the pressure on the surface of the elliptic cylinder. Now, the pressure p at any point in the fluid is given by (4), together with the expression for ϕ, given by (8). Since, however, $\partial\phi/\partial x \to 0$ as $\xi \to \infty$, p represents the pressure relative to that at infinity. Thus, if we denote the absolute pressure at any point in the fluid by p also and the corresponding pressure at infinity by p_∞, we can write

$$p-p_\infty = \rho U\frac{\partial\phi}{\partial x}.$$

In particular, if we denote the pressure on the surface of the cylinder by p_s, we have

$$\frac{p_s-p_\infty}{\frac{1}{2}\rho U^2} = \frac{2}{U}\left(\frac{\partial\phi}{\partial x}\right)_{\xi=\xi_0}.$$

Substituting the expression (8) for ϕ, we find

$$\frac{p_s-p_\infty}{\frac{1}{2}\rho U^2} = \frac{1}{h_0^2}\sum_{n=0}^{\infty}\alpha_n\{e^{-(n-1)\xi_0}\cos(n+1)\eta-e^{-(n+1)\xi_0}\cos(n-1)\eta\}, \tag{44}$$

where $h_0^2 = \cosh^2\xi_0-\cos^2\eta$, and this gives the general expression for the pressure on the surface of the elliptic cylinder whose major axis is parallel to the stream.

We have obtained, likewise, the general expression for the pressure distribution on the surface of the elliptic cylinder placed perpendicularly to the stream, and this will be given in the Appendix.

It is of interest to compute the pressure distributions on the surface of an elliptic cylinder whose major axis is parallel to the stream. Using the above expression (44), computations of the values of the pressure coefficient $(p_s-p_\infty)/(\frac{1}{2}\rho U^2)$ have been carried out for the two cases when $R = 4{\cdot}8$, $b/a = \frac{1}{2}$ and $R = 3{\cdot}2$, $b/a = \frac{1}{4}$ respectively. The results are given in Table III and are shown in Fig. 4.† From this figure it is seen that when the Reynolds number is small the pressure gradient is inappreciable for the most part of the surface of the cylinder except in the neighbourhood of the leading edge where a very sharp pressure gradient appears.

† In Table III the values of the pressure coefficient at $x/a = 1{\cdot}0$ have been omitted, since the series on the right-hand side of (44) is slowly convergent in the neighbourhood of the trailing edge of the cylinder. In Fig. 4 the curves near the trailing edge have been drawn by extrapolation.

TABLE III. *Values of* $(p_s - p_\infty)/(\frac{1}{2}\rho U^2)$

x/a	$R = 4{\cdot}8$, $b/a = \frac{1}{2}$	$R = 3{\cdot}2$, $b/a = \frac{1}{4}$
$-1{\cdot}0$	$3{\cdot}90$	$7{\cdot}67$
$-0{\cdot}9$	$1{\cdot}72$	$1{\cdot}03$
$-0{\cdot}8$	$0{\cdot}90$	$0{\cdot}33$
$-0{\cdot}6$	$0{\cdot}09$	$-0{\cdot}14$
$-0{\cdot}4$	$-0{\cdot}37$	$-0{\cdot}37$
$-0{\cdot}2$	$-0{\cdot}69$	$-0{\cdot}53$
0	$-0{\cdot}87$	$-0{\cdot}64$
$0{\cdot}2$	$-0{\cdot}91$	$-0{\cdot}68$
$0{\cdot}4$	$-0{\cdot}84$	$-0{\cdot}69$
$0{\cdot}6$	$-0{\cdot}75$	$-0{\cdot}68$
$0{\cdot}8$	$-0{\cdot}71$	$-0{\cdot}74$
$1{\cdot}0$	—	—

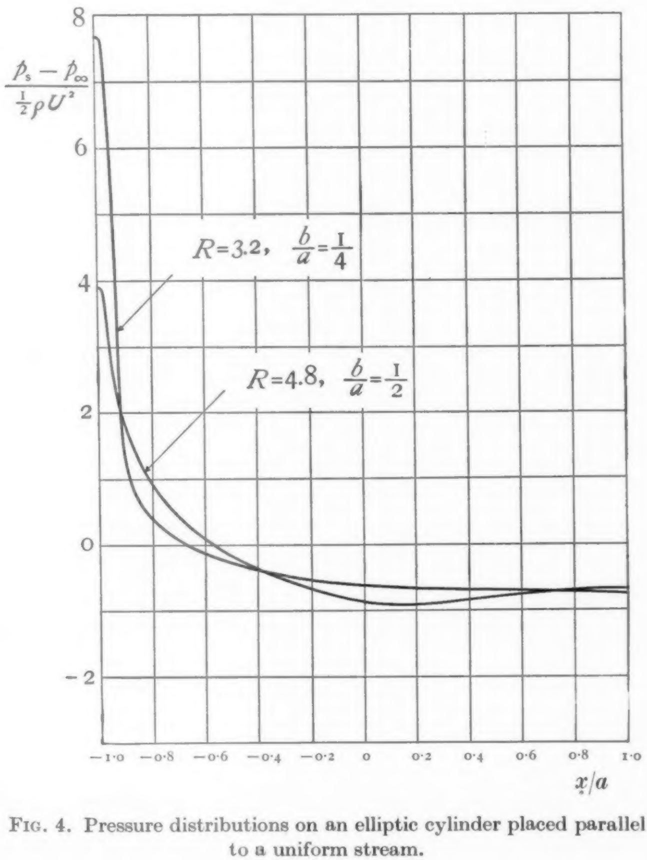

FIG. 4. Pressure distributions on an elliptic cylinder placed parallel to a uniform stream.

APPENDIX

For completeness, we shall give here, as an addendum, various results for the case of an elliptic cylinder whose major axis, of length $2a$, is at right angles to the uniform stream of velocity U. For brevity, only relevant parts of the analysis will be given here.

1. The general solution of Oseen's equations

The fundamental solution of Oseen's equations is given by ϕ and χ which satisfy equations (5) and (6) respectively. In this case, however, it is convenient to introduce the elliptic coordinates (ξ, η) defined by

$$x+iy = c\sinh(\xi+i\eta),$$

where $c = \sqrt{(a^2-b^2)}$.

The appropriate expression for the function ϕ is given by

$$\phi = cU\alpha_0^*\xi - cU\sum_{n=1}^{\infty}\frac{1}{n}\alpha_n^* e^{-n\xi}\cos n\eta,$$

where the α_n^* are constants of integration.† The appropriate expression for the function χ is given by

$$\chi = Ue^{kc\sinh\xi\cos\eta}\sum_{m=0}^{\infty}\beta_m^*\,\mathrm{FEK}_m^*(\xi)\mathrm{ce}_m^*(\eta),$$

where $k = U/2\nu$ and the β_m^* are also constants of integration. Here, the function $\mathrm{ce}_m^*(\eta)$ is the Mathieu function, namely

$$\mathrm{ce}_{2m}^*(\eta) = \sum_{r=0}^{\infty} A_{2r}^{(2m)}\cos 2r\eta,$$

$$\mathrm{ce}_{2m+1}^*(\eta) = \sum_{r=0}^{\infty} A_{2r+1}^{(2m+1)}\cos(2r+1)\eta,$$

while the function $\mathrm{FEK}_m^*(\xi)$ is the modified Mathieu function associated with $\mathrm{ce}_m^*(\eta)$ and is defined as

$$\mathrm{FEK}_{2m}^*(\xi) = \frac{1}{A_0^{(2m)}}\sum_{r=0}^{\infty} A_{2r}^{(2m)}K_r(\tfrac{1}{2}\kappa e^{\xi})I_r(\tfrac{1}{2}\kappa e^{-\xi}),$$

$$\mathrm{FEK}_{2m+1}^*(\xi) = \frac{1}{A_1^{(2m+1)}}\sum_{r=0}^{\infty} A_{2r+1}^{(2m+1)}\{K_{r+1}(\tfrac{1}{2}\kappa e^{\xi})I_r(\tfrac{1}{2}\kappa e^{-\xi})+K_r(\tfrac{1}{2}\kappa e^{\xi})I_{r+1}(\tfrac{1}{2}\kappa e^{-\xi})\}.$$

The constants of integration α_n^* and β_m^* in the expressions for ϕ and χ can be determined by the boundary conditions at the surface of the cylinder which is defined by $\xi = \xi_0$. Thus we have

$$\alpha_0^* = -\frac{1}{2\kappa}\sum_{m=0}^{\infty}\mathrm{ce}_m^*(0)\beta_m^*,$$

$$\alpha_n^* e^{-n\xi_0} - \sum_{m=0}^{\infty}\beta_m^* F_{m,n}^*(\xi_0) = \begin{cases}\cosh\xi_0 & (n=1),\\ 0 & (n=2,3,\ldots),\end{cases}$$

† The various quantities in this case are distinguished from those previously given by adding an asterisk.

and
$$\alpha_n^* e^{-n\xi_0} - \sum_{m=0}^{\infty} \beta_m^* G_{m,n}^*(\xi_0) = \begin{cases} -\sinh\xi_0 & (n=1), \\ 0 & (n=2,3,\ldots), \end{cases}$$
where
$$F_{m,n}^*(\xi) = \frac{1}{\kappa}\{\mathrm{FEK}_m^{*\prime}(\xi) I_{m,n}^*(\xi) - \mathrm{FEK}_m^*(\xi) I_{m,n}^{*\prime}(\xi)\},$$
$$G_{m,n}^*(\xi) = \frac{1}{\kappa}\mathrm{FEK}_m^*(\xi)\{\kappa\sinh\xi\, I_{m,n-1}^*(\xi) - n I_{m,n}^*(\xi) - \kappa\sinh\xi\, I_{m,n+1}^*(\xi)\},$$
with
$$I_{2m,n}^*(\xi) = \tfrac{1}{2}\sum_{r=0}^{\infty} A_{2r}^{(2m)}\{I_{n-2r}(\kappa\sinh\xi) + I_{n+2r}(\kappa\sinh\xi)\},$$
$$I_{2m+1,n}^*(\xi) = \tfrac{1}{2}\sum_{r=0}^{\infty} A_{2r+1}^{(2m+1)}\{I_{n-2r-1}(\kappa\sinh\xi) + I_{n+2r+1}(\kappa\sinh\xi)\};$$
as before, κ stands for kc.

Further, the β_m^* are determined by solving the following system of simultaneous algebraic linear equations:
$$\sum_{m=0}^{\infty} \beta_m^*\{F_{m,n}^*(\xi_0) - G_{m,n}^*(\xi_0)\} = \begin{cases} -e^{\xi_0} & (n=1), \\ 0 & (n=2,3,\ldots). \end{cases}$$

2. The general expression for the drag coefficient

By introducing the drag coefficient $C_D = D/(\rho U^2 . 2a)$, we get
$$C_D = \frac{2\pi}{R}\sum_{m=0}^{\infty} \mathrm{ce}_m^*(0)\beta_m^*,$$
where $R = 2aU/\nu$ is the Reynolds number.

3. The general expression for the stream function

The stream function, expressed in the non-dimensional form, is given by $\psi_1 = \psi/(Ua)$, where
$$\psi_1 = \operatorname{sech}\xi_0(\cosh\xi - e^{\xi_0-\xi}\cosh\xi_0)\sin\eta + $$
$$+\operatorname{sech}\xi_0 \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \beta_m^*\{F_{m,n}^*(\xi) - e^{n(\xi_0-\xi)}F_{m,n}^*(\xi_0)\}\frac{\sin n\eta}{n}.$$

4. The general expression for the pressure coefficient

The pressure distribution on the surface of the cylinder can be computed from the equation
$$\frac{p_s - p_\infty}{\frac{1}{2}\rho U^2} = \frac{1}{h_0^{*2}}\sum_{n=0}^{\infty} \alpha_n^*\{e^{-(n-1)\xi_0}\cos(n+1)\eta + e^{-(n+1)\xi_0}\cos(n-1)\eta\},$$
where
$$h_0^{*2} = \sinh^2\xi_0 + \cos^2\eta.$$

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THE TRANSIENT MOTION OF SOUND WAVES IN TUBES

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SUMMARY

This paper falls into two parts; sections 2–5 deal with the transient motion of sound waves in an infinite tube of constant cross-section; while in sections 6–9 the effect of slight distortion of the cross-section is considered. In the first part a formula is obtained for the velocity potential by use of Heaviside's operators, and an asymptotic expression for this is then found by the method of steepest descent. In the second part of the paper a series solution is obtained for the velocity potential in a tube of slightly variable cross-section. The particular case of a tube of near rectangular cross-section is investigated, and for such a tube a longitudinal harmonic distortion is considered in detail.

1. Introduction

THE motion of sound waves along an infinitely long straight tube of constant cross-section, filled with a perfect gas, when the motion at every point is assumed to have a harmonic time dependence, is well known and was considered by Lord Rayleigh (**1**).

A slightly different problem is considered below. The gas is taken to be initially at rest in a state of equilibrium, and a known disturbance is assumed to commence at a particular cross-section of the tube. Linear approximations are used, the velocity potential is assumed to satisfy the wave equation and is taken to be continuous and have continuous derivatives everywhere; thus discontinuities such as shock waves are not considered. The velocity potential may be expanded in a series of eigenfunctions connected with the cross-sectional shape. Due to the linearity of the wave equation each such term in the expansion of the velocity potential may be considered separately.

An expression for the velocity potential valid at all points in the tube is obtained by the use of Heaviside's operational method. It contains an integral, however, which is difficult to evaluate. An asymptotic form of the velocity potential valid for large values of the time t is obtained from the operational form of the solution by use of the method of steepest descent. A harmonic time dependence of the forcing motion is considered in some detail.

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For this case the resulting field can be split into two parts, namely, the transient motion, and the steady state solution or main signal which is propagated or attenuated according as the frequency of the forcing field is greater or less than the critical frequency of the mode considered. The transient disturbance travels with the free space velocity of sound, while the main signal travels with the group velocity of the mode considered; this group velocity is less than the free space velocity.

For a propagated mode the method of steepest descent due to Debye cannot be used at sections of the tube near the front of the main signal, since at such points a pole of the integrand is near the saddle point. The method developed by Clemmow (**2**) can be used, however, and a solution obtained in terms of Fresnel integrals.

In the second half of the paper the effect of small variations in cross-section of the tube is considered. It is assumed that this distortion of the tube modifies only slightly the results obtained from the corresponding tube of constant cross-section. If terms of the second order of the variation in cross-section are neglected, the problem becomes that of determining a solution of the wave equation with given normal derivatives on a tube of constant cross-section. This is solved by means of a Green's function.

The prescribed disturbance at a particular cross-section may be split into a sum of modes. The resulting sound wave will also consist of a sum of modes, but energy will be continually transformed between the modes with passage along the tube. The modes are no longer independent but are coupled to each other by the distortion of the tube. The transfer of energy to any particular mode is assumed to be small. If the prescribed disturbance consists of a single mode, the effect of the variation in the section is to transfer some of the energy from that mode to other modes. Energy in any one mode, on passing through a distorted section of the tube, is partially transformed to other modes, which, if they have a cut-off frequency below the frequency of the disturbance, will be propagated with different group velocities from the original mode, or if their cut-off frequency is higher than the frequency of the disturbance, they will be attenuated.

If the variation in cross-section is assumed to be harmonic, resonance phenomena occur for certain values of the frequency of the original disturbance. For such frequencies a large amount of energy is transferred from the original mode to another mode, thus violating an assumption on which the theory is based. This difficulty does not arise, however, if signals of only small duration are considered, in which case the energy-transfer to any single mode does not become large.

I wish to thank Mr. D. S. Jones and Mr. E. Wild for their very helpful advice and suggestions.

2. Integral expression for the velocity potential

Consider an infinitely long straight tube of constant cross-section with rigid walls. The tube is filled with a perfect gas with constant specific heats. Any motion of the particles of the gas will be taken to be small, and the accompanying thermal changes to be adiabatic.

Let x, y, z be a right-handed system of cartesian coordinates such that the x-axis is parallel to the wall of the tube. The wall of the tube is denoted by σ, a normal cross-section of the tube by A, the bounding curve of A by C, and the unit outward-normal to C by $\boldsymbol{\nu}$.

The gas is initially in a uniform state of equilibrium, physical properties in this state being denoted by the suffix 0. At time $t = 0$ the gas is disturbed over the plane face of a normal cross-section of the tube, which will be taken to be the section $x = 0$, and it is required to determine the subsequent motion.

If terms of the second order of small quantities are neglected, the velocity potential ϕ satisfies the equation

$$\nabla^2\phi - \frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2} = 0, \tag{1}$$

where a is the free space velocity of sound in the gas. Having regard to the above remarks on the origin of the time scale, ϕ and $\partial\phi/\partial t$ are taken to be zero for $t = 0$.

The boundary condition at a rigid wall for a perfect gas requires that the normal velocity shall be zero; hence

$$\frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \sigma. \tag{2}$$

A disturbance at a normal cross-section is completely prescribed if ϕ is given at points on that cross-section. Thus a solution of (1) is required with given values of ϕ at $x = 0$ subject to condition (2) and a radiation condition at infinity.

The operator $\partial/\partial t$ will be denoted by p, the Heaviside operational representation of a function of $f(t)$ will be denoted by $f(p)$, and the symbol $\doteqdot$ will be taken to read as 'the operational representation of'.

If ϕ is prescribed at $x = 0$ for all positive values of the time, being zero for $t < 0$, subject to ϕ at $x = 0$ satisfying the condition on C stated above, then for points inside and on the curve C it is possible to expand ϕ in the

$$(\phi)_{x=0} = \sum_n a_n(t)\psi_n(y,z), \tag{3}$$

where

$$a_n(t) = \int_A (\phi)_{x=0}\psi_n^*(y,z)\,dS. \tag{4}$$

The set of eigenfunctions $\psi_n(y,z)$ are solutions of the equation

$$\frac{\partial^2\psi_n}{\partial y^2}+\frac{\partial^2\psi_n}{\partial z^2}+k_n^2\psi_n = 0 \quad \text{over } A,$$

with boundary condition

$$\frac{\partial\psi_n}{\partial\nu} = 0 \quad \text{on } C,$$

and $\psi_n^*(y,z)$ is the complex conjugate of $\psi_n(y,z)$. It can be shown that

$$\int\limits_A \psi_n\psi_m^*\,dS = 0 \quad (n \neq m),$$

and we choose ψ_n such that

$$\int\limits_A |\psi_n|^2\,dS = 1.$$

Equation (1) becomes, in operational form,

$$\nabla^2\bar{\phi}-\frac{p^2}{a^2}\bar{\phi} = 0, \tag{5}$$

a solution of which may be written

$$\bar{\phi} = \sum_n \bar{A}_n(x,p)\psi_n(y,z), \tag{6}$$

and the $\bar{A}_n$'s are to be determined.

On multiplying (5) by $\psi_n^*(y,z)$ and integrating over the area A it is seen that

$$\frac{\partial^2\bar{A}_n}{\partial x^2}-\left(k_n^2+\frac{p^2}{a^2}\right)\bar{A}_n = 0.$$

The general solution of the last equation is

$$\bar{A}_n(x,p) = \bar{\alpha}_n(p)e^{-\lambda_n x}+\bar{\beta}_n(p)e^{\lambda_n x}, \tag{7}$$

where

$$\lambda_n^2 = \frac{p^2}{a^2}+k_n^2.$$

Considered as a function of p, λ_n exists on a Riemann surface of two sheets with branch points at $p = \pm iak_n$ (points B and B' of Fig. 1) and branch line joining these two points along the imaginary axis. We take $\lambda_n(p)$ in (7) to lie on the sheet of the Riemann surface on which λ_n is real and positive for $p = 0$, this sheet being defined as the first sheet of the Riemann surface.

The first term on the right-hand side of (7) represents a wave travelling in the positive x-direction and the second a wave travelling in the negative x-direction. Thus since waves must diverge from their source, we must have

$$\bar{\alpha}_n(p) = 0 \quad \text{for } x < 0,$$

$$\bar{\beta}_n(p) = 0 \quad \text{for } x > 0,$$

while, at $x = 0$,

$$\bar{A}_n = \bar{a}_n.$$

It follows that, for $x > 0$,

$$\bar{\phi} = \sum_n \bar{a}_n(p) e^{-\lambda_n x} \psi_n(y,z), \tag{8}$$

and, for $x < 0$,
$$\bar{\phi} = \sum_n \bar{a}_n(p) e^{\lambda_n x} \psi_n(y,z).$$

Now it is known that

$$p e^{-\lambda_n x} \doteqdot \delta\left(t-\frac{x}{a}\right) - H\left(t-\frac{x}{a}\right) x k_n \frac{J_1\{k_n \sqrt{(a^2t^2-x^2)}\}}{\sqrt{(t^2-x^2/a^2)}}, \tag{9}$$

where $J_1(x)$ is the Bessel function of order 1, $\delta(t-x/a)$ the Dirac delta function, and $H(t-x/a)$ the Heaviside unit function.

Hence, by the operational representation of a Faltung integral, it is seen that

$$\phi = \sum_n H\left(t-\frac{x}{a}\right)\psi_n(y,z)\left\{a_n\left(t-\frac{x}{a}\right) - \int\limits_{x/a}^{t} \frac{a_n(t-y) x k_n J_1\{k_n \sqrt{(y^2a^2-x^2)}\}}{\sqrt{(y^2-x^2/a^2)}}\,dy\right\} \tag{10}$$

for $x > 0$, and a corresponding expression for ϕ exists for $x < 0$.

The integrals in the above expansion for ϕ are difficult to evaluate for most forms of $a_n(t)$, but an asymptotic form for ϕ may be obtained by the method of steepest descent.

3. Asymptotic form of the velocity potential

Since the modes are propagated independently of each other, only the propagation of a single mode need be considered, and thus at $x = 0$, ϕ is taken to be equal to $a_n(t)\psi_n(y,z)$.

In order to obtain an expression for ϕ valid for large t, the following integral will be considered,

$$I = \frac{1}{2\pi i} \int\limits_{d-i\infty}^{d+i\infty} \exp(pt - \lambda_n x) \frac{\bar{a}_n(p)}{p}\,dp, \tag{11}$$

where $d \geqslant 0$, $x \geqslant 0$, and $\bar{a}_n(p)/p$ is regular on the Riemann surface of $\lambda_n(p)$, and the path of integration lies on the first sheet of this surface.

By putting $x = cta$, where $0 \leqslant c < 1$, and regarding c as constant, (11) may be written

$$I = \frac{1}{2\pi i} \int\limits_{d-i\infty}^{d+i\infty} \exp\{tg(p)\} \frac{\bar{a}_n(p)}{p}\,dp, \tag{12}$$

where $g(p) = p - ca\lambda_n(p)$.

Cases for which $c > 1$ are not considered, since they correspond to points ahead of the disturbance. The integral (12) is in the form for which an asymptotic expression can be found by the method of steepest descent.

It is easily shown that $g(p)$ has saddle points at

$$p = \pm iak_n(1-c^2)^{-\frac{1}{2}}$$

(points S and S' of Fig. 1) on the first sheet of the Riemann surface of $\lambda_n(p)$.

The path of integration of I is deformed to pass through the saddle point and to lie along the line

$$\operatorname{im}(p-c\lambda_n) = \text{constant}.$$

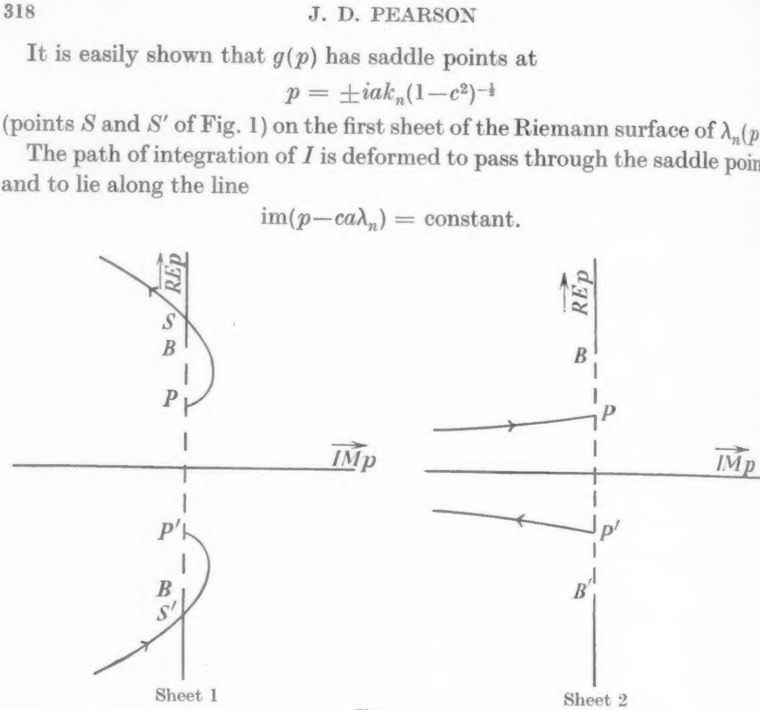

FIG. 1.

A diagram of the path on the two sheets of the Riemann surface is given in Fig. 1.

Applying the method of steepest descent, contributions are considered only from the portion of the integrand near to the saddle point, and Debye's formula gives for large t

$$I \sim \frac{1}{2\pi i}\times$$
$$\times\left\{-\left(\frac{2\pi}{t}\right)^{\frac{1}{2}}\frac{(k_n a)^{\frac{1}{2}}c}{iak_n(1-c^2)^{\frac{5}{4}}}\left[a_n\left(\frac{iak_n}{(1-c^2)^{\frac{1}{2}}}\right)e^{itk_n a\sqrt{(1-c^2)}}+a_n\left(\frac{-iak_n}{(1-c^2)^{\frac{1}{2}}}\right)e^{-itk_n a\sqrt{(1-c^2)}}\right]\right\}. \tag{13}$$

The expression (13) is valid only for large values of $|tg''\{iak_n(1-c^2)^{-\frac{1}{2}}\}|$ and thus cannot be used at a zero of $g''\{iak_n(1-c^2)^{-\frac{1}{2}}\}$, where accents denote derivatives. The asymptotic form becomes invalid, therefore, for values of c near $c = 1$, i.e. for points near the wave front.

If the original disturbance is taken to be such that at $x = 0$

$$\phi = a_n(t)\psi_n(y,z) \quad \text{for } t \geqslant 0,$$

then

$$\bar{\phi} = \bar{a}_n(p)e^{-\lambda_n x}\psi_n(y,z) \quad (x \geqslant 0),$$

which gives on inversion

$$\phi = \psi_n(y,z)\frac{1}{2\pi i}\int\limits_{d-i\infty}^{d+i\infty} \frac{\bar{a}_n(p)}{p} e^{tg(p)}\, dp, \tag{14}$$

which is of the form of the integral (11).

In order to apply the formula (13) it is necessary that $a_n(p)/p$ is regular on the Riemann surface of $\lambda_n(p)$. If $a_n(p)/p$ has poles, then contributions from these must be considered.

4. Harmonic forcing motion

Now consider the forcing term

$$a_n(t) = H(t)\cos\omega t,$$

then

$$a_n(p) = \frac{p^2}{p^2+\omega^2}.$$

Substitution of this expression for $a_n(p)/p$ in (14) gives rise to an integral which is formally similar to that of expression (11). The integrand, however, has poles at $p = \pm i\omega$. On deforming the path of integration to lie along the line of steepest descent, contributions to the integral may arise from these poles. As is seen in Fig. 1, this is the case if

$$\omega > k_n a(1-c^2)^{-\frac{1}{2}},$$

which implies $\omega > k_n a$ and $t > x\omega/a(\omega^2-a^2k_n^2)^{\frac{1}{2}}$ and a contribution

$$H\left(t-\frac{x}{a}\frac{\omega}{\sqrt{(\omega^2-a^2k_n^2)}}\right)\psi_n(y,z)\cos\left(\omega t-\frac{x}{a}(\omega^2-k_n^2\,a^2)^{\frac{1}{2}}\right)$$

must be included in our asymptotic form of (14) as well as the term given by (13). It is easily shown that the path of steepest descent crosses the branch line of the Riemann surface at the points given by

$$p = \pm iak_n(1-c^2)^{\frac{1}{2}}$$

(the points P and P' of Fig. 1). Hence if $\omega < k_n a(1-c^2)^{\frac{1}{2}}$, which implies that $\omega < k_n a$ and $t > xk_n(k_n^2 a^2-\omega^2)^{\frac{1}{2}}$, a contribution

$$H\left(t-\frac{xk_n}{a(k_n^2-\omega^2/a^2)^{\frac{1}{2}}}\right)\psi_n(y,z)\exp\left(-\frac{x}{a}(k_n^2\,a^2-\omega^2)^{\frac{1}{2}}\right)\cos\omega t$$

arises from the poles at $p = \pm i\omega$, for $x > 0$.

The asymptotic solution for ϕ such that

$$\phi = H(t)\cos\omega t\,\psi_n(y,z) \quad \text{at } x = 0$$

is for $n > 0$ given by

$$\phi \sim \left[\left(\frac{2}{\pi k_n at}\right)^{\frac{1}{2}} H\left(t-\frac{x}{a}\right)\frac{ck_n^2\,a^2\cos\{tk_n\,a(1-c^2)^{\frac{1}{2}}-\frac{1}{4}\pi\}}{(1-c^2)^{\frac{1}{2}}\{k_n^2\,a^2-\omega^2(1-c^2)\}}+\phi'\right]\psi_n \tag{15}$$

for $x > 0$, or

$$\phi \sim \left[-\left(\frac{2}{\pi k_n a}\right)^{\frac{1}{2}} H\left(t-\frac{x}{a}\right) \frac{xtak_n^2 \cos\{k_n(t^2a^2-x^2)^{\frac{1}{2}}-\frac{1}{4}\pi\}}{(t^2-x^2/a^2)^{\frac{1}{4}}\{t^2(\omega^2-k_n^2 a^2)-x^2\omega^2/a^2\}}+\phi'\right]\psi_n,$$

where $$\phi' = H\left(t-\frac{x\omega}{a(\omega^2-a^2k_n^2)^{\frac{1}{2}}}\right)\cos\left(\omega t-\frac{x}{a}(\omega^2-k_n^2 a^2)^{\frac{1}{2}}\right)$$

if $\omega > k_n a$ and

$$\phi' = H\left(t-\frac{xk_n}{a(k_n^2-\omega^2/a^2)^{\frac{1}{2}}}\right)\exp\left(-\frac{x}{a}(k_n^2 a^2-\omega^2)^{\frac{1}{2}}\right)\cos\omega t$$

if $\omega < k_n a$. For the case $\omega > k_n a$, ϕ' will be called the main signal term.

If $n = 0$ contributions to ϕ arise only from the poles at $p = \pm i\omega$ to give

$$\phi = H\left(t-\frac{x}{a}\right)\psi_0 \cos\left(\omega t-\frac{\omega x}{a}\right).$$

Thus, a disturbance which is constant over the cross-section is propagated unchanged.

The velocity of the disturbance represented by ϕ' is seen to be the group velocity of the nth mode.

The transient disturbance represented by the first term on the right-hand side of (15) takes infinite values when $c = 1$ and when $c = (1-a^2k_n^2/\omega^2)^{\frac{1}{2}}$, which correspond to cross-sections at the front of the disturbance and at the front of the main signal, respectively. Thus in the neighbourhood of these sections the asymptotic form (15) becomes invalid. For values of c near the front of the main signal, the pole at $p = i\omega$ and the saddle point at $p = iak_n(1-c^2)^{-\frac{1}{2}}$ are close to one another and (15) tends to infinity.

At a point near the front of the main signal, Debye's formula may be applied if t is taken sufficiently large, but the formula breaks down at the actual main signal front. An asymptotic form which is finite and continuous at the main signal front is obtained in section 5. The transient is seen to oscillate about zero with decreasing amplitude and increasing wavelength, for fixed t, as x tends to zero, or for fixed x, as t tends to infinity.

For general prescribed value of ϕ across a normal cross-section containing all the modes, with a time dependence $H(t)\cos\omega t$, it is seen that all modes are attenuated except the modes with critical frequency smaller than ω.

5. Approximation near the front of the main signal

In order to obtain an asymptotic expression for ϕ which is continuous at the front of the main signal, the following integral will be considered,

$$I = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \frac{p}{p^2+\omega^2}\exp\{tg(p)\}\,dp, \tag{16}$$

where $\omega > k_n a$. This integral can be written

$$I = \frac{1}{4\pi i}\left(\int\limits_{d}^{d+i\infty} \frac{\exp\{tg(p)\}}{p-i\omega}\,dp + \int\limits_{d-i\infty}^{d} \frac{\exp\{tg(p)\}}{p+i\omega} + \right.$$

$$\left. + \int\limits_{d}^{d+i\infty} \frac{\exp\{tg(p)\}}{p+i\omega}\,dp + \int\limits_{d-i\infty}^{d} \frac{\exp\{tg(p)\}}{p-i\omega}\,dp\right). \qquad (17)$$

The third and fourth integrals in (17) have integrals which are regular on the Riemann surface of $\lambda_n(p)$ in the range of integration. Debye's method of steepest descent can be applied to these integrals and their sum is asymptotically equal to

$$\left(\frac{k_n a}{2\pi t}\right)^{\frac{1}{2}} \frac{c}{(1-c^2)^{\frac{1}{4}}\{k_n a+\omega(1-c^2)^{\frac{1}{2}}\}} \cos\{tk_n a(1-c^2)^{\frac{1}{2}}-\tfrac{1}{4}\pi\}. \qquad (18)$$

The first and second integrals in equation (17) will be evaluated by an amended form of the method of steepest descent.

Consider the integral

$$I_1 = \int\limits_{d}^{d+i\infty} \frac{\exp\{tg(p)\}}{p-i\omega}\,dp. \qquad (19)$$

The path of integration is deformed to lie along the line of steepest descent, the contribution from the pole at $p = i\omega$ being dealt with as in section 4. It can be shown by methods similar to those used in the proof of Watson's lemma, that, neglecting contributions from the pole, I_1 defined by equation (19), is asymptotically equal to

$$I_1 \sim e^{tg(p_0)} \int\limits_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}tz^2}\,dz}{z+\{-g''(p_0)\}^{\frac{1}{2}}(p_0-i\omega)}, \qquad (20)$$

where $p_0 = ik_n a(1-c^2)^{-\frac{1}{2}}$.

The function D is defined to be equal to

$$(|p_0|-\omega)(1-c^2)^{\frac{3}{4}}/c(k_n a)^{\frac{1}{2}}$$

and thus

$$(p_0-i\omega)\{-g''(p_0)\}^{\frac{1}{2}} = -iDe^{\frac{1}{4}i\pi}.$$

Equation (20) may be rewritten

$$I_1 \sim \int\limits_{-\infty}^{\infty} \frac{iD}{z^2+D^2 i} \exp\{tg(p_0)+\tfrac{1}{4}i\pi-\tfrac{1}{2}tz^2\}\,dz.$$

Now

$$\frac{1}{z^2+D^2 i} = \int\limits_{0}^{\infty} \exp\{-x(z^2+D^2 i)\}\,dx$$

if

$$z^2+D^2 i \neq 0,$$

and thus

$$I_1 \sim \int_{-\infty}^{\infty}\int_{0}^{\infty} iD\exp\{tg(p_0)-\tfrac{1}{2}tz^2-x(z^2+D^2i)+\tfrac{1}{4}i\pi\}\,dxdz.$$

In the analysis below the order of integration is reversed and some justification of this is required. The order of integration may be reversed if the double integral of the modulus of the integrand is convergent. It can easily be shown that the double integral

$$\int_{-\infty}^{\infty}\int_{0}^{X} |D\exp\{tg(p_0)-\tfrac{1}{2}tz^2-x(z^2+D^2i)+\tfrac{1}{4}i\pi\}|\,dxdz$$

is convergent for finite X. The integral

$$R(X) = \int_{-\infty}^{\infty}\int_{X}^{\infty} \exp\{-\tfrac{1}{2}tz^2-x(z^2+D^2i)\}\,dxdz$$

will now be considered. Carrying out the integration with respect to x in $R(X)$, it is easily seen that

$$R(X) = \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}tz^2-X(z^2+D^2i)\}\frac{1}{z^2+D^2i}\,dz$$

or
$$R(X) = \left[\int_{-\infty}^{-\epsilon}+\int_{-\epsilon}^{\epsilon}+\int_{\epsilon}^{\infty}\right]\frac{\exp\{-\tfrac{1}{2}tz^2-X(z^2+D^2i)\}}{z^2+D^2i}\,dz,$$

where ϵ is positive.

Now

$$\left|\int_{-\epsilon}^{\epsilon}\frac{e^{-\frac{1}{2}tz^2-X(z^2+D^2i)}}{z^2+D^2i}\,dz\right| \leqslant \frac{1}{D^2}\int_{-\epsilon}^{\epsilon} e^{-(\frac{1}{2}t+X)z^2}\,dz$$
$$< \frac{2\epsilon}{D^2},$$

and

$$\left|\int_{\epsilon}^{\infty}\frac{e^{-\frac{1}{2}tz^2-X(z^2+D^2i)}}{z^2+D^2i}\,dz\right| \leqslant \frac{1}{D^2}\int_{\epsilon}^{\infty} e^{-(\frac{1}{2}t+X)z^2}\,dz$$
$$\leqslant \frac{1}{D^2}e^{-X\epsilon^2}\int_{0}^{\infty} e^{-\frac{1}{2}tz^2}\,dz.$$

We choose ϵ such that $X\epsilon^2$ is large, say $\epsilon = X^{-\frac{1}{3}}$, and thus $R(X) = o(1)$ as $X \to \infty$. Thus

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-\frac{1}{2}tz^2-x(z^2+D^2i)}\,dxdz = \int_{0}^{X}\int_{-\infty}^{\infty} e^{-\frac{1}{2}tz^2-x(z^2+D^2i)}\,dzdx+o\,(1)$$

and taking the limit as $X \to \infty$ we have

$$I_1 \sim \int_0^\infty \int_{-\infty}^\infty iD \exp\{tg(p_0)+\tfrac{1}{4}i\pi-\tfrac{1}{2}tz^2-x(z^2+D^2i)\}\, dzdx.$$

Carrying out the integration with respect to z in the last integral, we have

$$I_1 \sim \int_0^\infty \exp\{tg(p_0)+\tfrac{1}{4}i\pi-ixD^2\} \frac{i\pi^{\frac{1}{2}}D}{(x+\frac{1}{2}t)^{\frac{1}{2}}}\, dx.$$

Putting

$$(x+\tfrac{1}{2}t)^{\frac{1}{2}} = \frac{y}{|D|}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}$$

the last integral becomes

$$I_1 \sim \int_{|D|\sqrt{(t/\pi)}}^\infty \exp\{tg(p_0)+\tfrac{1}{4}i\pi-\tfrac{1}{2}i\pi y^2-\tfrac{1}{2}tiD^2\} \frac{i\pi D\, 2^{\frac{1}{2}}}{|D|}\, dy. \tag{21}$$

The Fresnel integral $\mathscr{E}(x)$ is defined to be

$$\mathscr{E}(x) = \int_x^\infty \exp(\tfrac{1}{2}y^2\pi i)\, dy \tag{22}$$

and $\mathscr{E}^*(x)$ is the complex conjugate of $\mathscr{E}(x)$. Equation (21) becomes, on using (22) and including the contribution from the pole, which has been omitted in the above analysis,

$$I_1 \sim 2\pi i \exp\{i\omega t-(ix/a)(\omega^2-k_n^2 a^2)^{\frac{1}{2}}\}- \\ -2^{\frac{1}{2}}\pi i\mathscr{E}^*\{|D|(t/\pi)^{\frac{1}{2}}\}\exp\{tg(p_0)+\tfrac{1}{4}i\pi-\tfrac{1}{2}tiD^2\} \tag{23}$$

for $\omega > |p_0|$ and

$$I_1 \sim 2^{\frac{1}{2}}\pi i\mathscr{E}^*\{|D|(t/\pi)^{\frac{1}{2}}\}\exp\{tg(p_0)+\tfrac{1}{4}i\pi-\tfrac{1}{2}tiD^2\}$$

for $\omega < |p_0|$.

The transformation from equation (20) to the form (23) may also be carried out by the use of the definition of the confluent hypergeometric function (**2**).

By a method similar to the above it can be shown that the integral I_2,

$$I_2 = \int_{d-i\infty}^{d} \frac{\exp\{tg(p)\}}{p+i\omega}\, dp,$$

has the asymptotic form

$$I_2 \sim 2\pi i \exp\{-i\omega t+(ix/a)(\omega^2-k_n^2 a^2)^{\frac{1}{2}}\}- \\ -2^{\frac{1}{2}}\pi i\mathscr{E}\{|D|(t/\pi)^{\frac{1}{2}}\}\exp\{-tg(p_0)-\tfrac{1}{4}i\pi+\tfrac{1}{2}itD^2\}$$

for $\omega > |p_0|$ and

$$I_2 \sim 2^{\frac{1}{2}}\pi i\mathscr{E}\{|D|(t/\pi)^{\frac{1}{2}}\}\exp\{-tg(p_0)-\tfrac{1}{4}i\pi+\tfrac{1}{2}itD^2\}$$

for $\omega < |p_0|$.

A few properties of the Fresnel integral defined by (22) will now be given (**4**).

It can be shown that

$$\mathscr{E}(x) = \{B(x)+iA(x)\}e^{\frac{1}{2}\pi i x^2},$$

where
$$B(x) = \frac{1}{\pi 2^{\frac{1}{2}}}\int\limits_0^\infty e^{-\frac{1}{2}\pi x^2 y}\frac{y^{\frac{1}{2}}}{1+y^2}\,dy$$

and
$$A(x) = \frac{1}{\pi 2^{\frac{1}{2}}}\int\limits_0^\infty e^{-\frac{1}{2}\pi x^2 y}\frac{y^{-\frac{1}{2}}}{1+y^2}\,dy.$$

For large values of x

$$A(x) \sim \frac{1}{\pi x}, \qquad B(x) = o\left(\frac{1}{x}\right).$$

We define the angle χ such that

$$\chi(u) = \arg\{B(u)+iA(u)\}.$$

Tables of $A(u)$ and $B(u)$ are given by Rankin (**4**).

If at $x = 0$ $\qquad \phi = H(t)\cos\omega t\,\psi_n(y,z) \quad (\omega > k_n a)$

then for $x > 0$ $\qquad \phi = \psi_n(y,z)I,$

where I is the integral (16). Thus

$$\phi \sim \psi_n(y,z)\left\{2^{-\frac{1}{2}}|\mathscr{E}\{|D|(t/\pi)^{\frac{1}{2}}\}|\cos[tk_n a(1-c^2)^{\frac{1}{2}}-\chi\{|D|(t/\pi)^{\frac{1}{2}}\}+\tfrac{1}{4}\pi]+\right.$$
$$\left.+\frac{1}{\sqrt{2\pi t}}\frac{(k_n a)^{\frac{1}{2}}c\cos\{tk_n a(1-c^2)^{\frac{1}{2}}-\frac{1}{4}\pi\}}{\{k_n a+\omega(1-c^2)^{\frac{1}{2}}\}(1-c^2)^{\frac{1}{4}}}\right\} \tag{24}$$

for $\omega < k_n a(1-c^2)^{-\frac{1}{2}}$, and

$$\phi \sim \psi_n(y,z)\left\{-2^{-\frac{1}{2}}|\mathscr{E}\{|D|(t/\pi)^{\frac{1}{2}}\}|\cos[tk_n a(1-c^2)^{\frac{1}{2}}-\chi\{|D|(t/\pi)^{\frac{1}{2}}\}+\tfrac{1}{4}\pi]+\right.$$
$$\left.+\frac{1}{\sqrt{2\pi t}}\frac{(k_n a)^{\frac{1}{2}}c\cos\{tk_n a(1-c^2)^{\frac{1}{2}}-\frac{1}{4}\pi\}}{\{k_n a+\omega(1-c^2)^{\frac{1}{2}}\}(1-c^2)^{\frac{1}{4}}}+\cos\{\omega t-(x/a)(\omega^2-k_n^2 a^2)^{\frac{1}{2}}\}\right\} \tag{25}$$

for $\omega > k_n a(1-c^2)^{-\frac{1}{2}}$.

It should be noted that at the front of the main signal expressions (24) and (25) are equal. Thus the asymptotic expression obtained above is continuous at the front of the main signal: it is therefore of greater value than the expression (15) obtained previously, for values of c corresponding to sections near the front of the main signal.

For large t
$$|\mathscr{E}\{|D|(t/\pi)^{\frac{1}{2}}\}| \sim \frac{1}{|D|(t\pi)^{\frac{1}{2}}}$$

and thus the expression (24) and the first two terms of expression (25) become for large t equal to

$$\psi_n(y,z)\left(\frac{2k_n a}{\pi t}\right)^{\frac{1}{2}} \frac{ck_n a}{\{(k_n a)^2-\omega^2(1-c^2)\}(1-c^2)^{\frac{1}{4}}} \cos\{tk_n a(1-c^2)^{\frac{1}{2}}-\tfrac{1}{4}\pi\}.$$

Thus the expressions (24), (25) reduce to the values of ϕ obtained by Debye's method of steepest descent if t is taken sufficiently large.

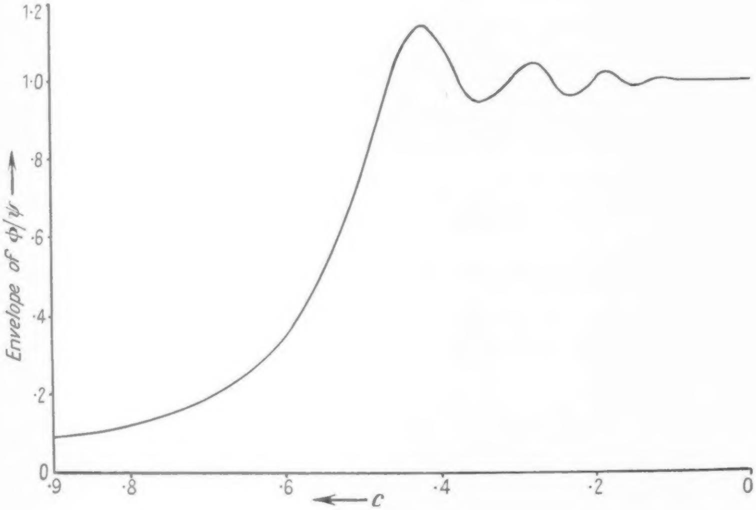

FIG. 2.

A graph of the amplitude of $\phi/\psi_n(y,z)$ is given in Fig. 2 for particular values of t, k_n, and ω. For points far from the main signal front the amplitude of $\phi/\psi_n(y,z)$ was calculated from (15), and for points near the main signal front expressions (24), (25) were used. The values of t, k_n, and ω used were

$$t = 500\pi/a \text{ sec.} \qquad k_n = 0{\cdot}1 \text{ cm.}^{-1} \qquad \omega = 4{,}000 \text{ cycles/sec.}$$

6. Tube of variable cross-section

The propagation of sound waves along a tube of slightly variable cross-section will now be considered. As before, ϕ is prescribed at $x = 0$ for all $t \geqslant 0$, and the form of the resulting sound wave is to be found.

The tube of constant cross-section, the propagation in which has previously been considered, will be described as tube 1, and a tube of variable cross-section whose cross-section for any x differs but slightly from that of

tube 1 will be denoted as tube 2. It is assumed that the cross-sections of tubes 1 and 2 are the same at $x = 0$. The velocity potential in tube 1 will be denoted by ϕ_1, and in tube 2 by ϕ_2; ϕ_1 and ϕ_2 are assumed to differ but slightly from one another. If the mathematical solutions obtained do not agree with this assumption, the method will automatically be invalid.

The bounding curve of tube 1 is assumed to be continuous and possess piece-wise continuous first derivatives. It is assumed to consist of N continuous curves whose equations may be written

$$f_r(y,z) = 0 \quad (r = 1, 2,..., N).$$

It is assumed that the variation in section is such that the bounding curve of cross-sections of tube 2 consists of N continuous curves whose equation may be written

$$f_r(y,z)+\epsilon F_r(x,y,z) = 0 \quad (r = 1, 2,..., N) \tag{26}$$

or
$$f_r\{y+\epsilon Y_r(x,y,z),\, z+\epsilon Z_r(x,y,z)\} = 0 \quad (r = 1, 2,..., N), \tag{27}$$

where ϵ is small. It is assumed that F_r, Y_r, Z_r and their derivatives are bounded. Thus equations (26), (27) are equivalent when terms of $O(\epsilon^2)$ are neglected, if

$$F_r(x,y,z) = Y_r\frac{\partial f_r}{\partial y}+Z_r\frac{\partial f_r}{\partial z} \quad (r = 1, 2,..., N). \tag{28}$$

Terms of $O(\epsilon^2)$ will be neglected throughout and the value of ϕ accurate to $O(\epsilon)$ will be determined; ϕ_1 and ϕ_2 are taken to be equal at $x = 0$ for all positive time, and it is assumed that

$$\phi_2 = \phi_1+\epsilon\Phi, \tag{29}$$

where $\epsilon\Phi$ is $O(\epsilon)$ and terms of $O(\epsilon^2)$ are neglected.

The boundary condition on the wall of tube 2 is that the normal derivative of ϕ_2 shall be zero. If l_r, m_r, n_r are the direction cosines of the outward normal to the rth section of the bounding curve of a normal cross-section, then from equation (26) it is seen that

$$l_r = \epsilon\frac{\partial F_r}{\partial x}, \qquad m_r = \epsilon\frac{\partial F_r}{\partial y}+\frac{\partial f_r}{\partial y}, \qquad n_r = \epsilon\frac{\partial F_r}{\partial z}+\frac{\partial f_r}{\partial z}. \tag{30}$$

The boundary condition on the wall of tube 2 may be written

$$l_r\frac{\partial \phi_2}{\partial x}+m_r\frac{\partial \phi_2}{\partial y}+n_r\frac{\partial \phi_2}{\partial z} = 0.$$

Substituting from equation (30) in the last equation, we have

$$\frac{\partial f_r}{\partial y}\frac{\partial \phi_2}{\partial y}+\frac{\partial f_r}{\partial z}\frac{\partial \phi_2}{\partial z}+\epsilon\left[\frac{\partial F_r}{\partial x}\frac{\partial \phi_2}{\partial x}+\frac{\partial F_r}{\partial y}\frac{\partial \phi_2}{\partial y}+\frac{\partial F_r}{\partial z}\frac{\partial \phi_2}{\partial z}\right]=0. \tag{31}$$

Thus from equations (29), (31) the boundary condition on the curve whose equation is $f_r(y,z)+\epsilon F_r(y,x,z)=0$ is

$$\frac{\partial f_r}{\partial y}\frac{\partial \phi_1}{\partial y}+\frac{\partial f_r}{\partial z}\frac{\partial \phi_1}{\partial z}+\epsilon\left[\frac{\partial F_r}{\partial x}\frac{\partial \phi_1}{\partial x}+\frac{\partial F_r}{\partial y}\frac{\partial \phi_1}{\partial y}+\frac{\partial F_r}{\partial z}\frac{\partial \phi_1}{\partial z}+\frac{\partial f_r}{\partial y}\frac{\partial \Phi}{\partial y}+\frac{\partial f_r}{\partial z}\frac{\partial \Phi}{\partial z}\right]=0$$

$$(r=1,2,\ldots,N). \tag{32}$$

Expanding $\partial f_r/\partial y$, $\partial f_r/\partial z$, $\partial \phi_1/\partial y$, $\partial \phi_1/\partial z$ in Taylor series and neglecting terms of $O(\epsilon^2)$ the boundary condition (32) may be expressed as a boundary condition on $f_r(y,z)=0$. Thus on $f_r(y,z)=0$, having regard to the fact that

$$\frac{\partial f_r}{\partial y}\frac{\partial \phi_1}{\partial y}+\frac{\partial f_r}{\partial z}\frac{\partial \phi_1}{\partial z}=0$$

on this curve, it is seen that

$$-\frac{\partial \Phi}{\partial \nu_r}=\left(\frac{\partial^2 f_r}{\partial y^2}Y_r+\frac{\partial^2 f_r}{\partial z\partial y}Z_r\right)\frac{\partial \phi_1}{\partial y}+\left(\frac{\partial^2 f_r}{\partial z^2}Z_r+\frac{\partial^2 f_r}{\partial y\partial z}Y_r\right)\frac{\partial \phi_1}{\partial z}+$$

$$+\frac{\partial f_r}{\partial y}\left(\frac{\partial^2 \phi_1}{\partial y^2}Y_r+\frac{\partial^2 \phi_1}{\partial z\partial y}Z_r\right)+\frac{\partial f_r}{\partial z}\left(\frac{\partial^2 \phi_1}{\partial z^2}Z_r+\frac{\partial^2 \phi_1}{\partial y\partial z}Y_r\right)+$$

$$+\frac{\partial F_r}{\partial x}\frac{\partial \phi_1}{\partial x}+\frac{\partial F_r}{\partial y}\frac{\partial \phi_1}{\partial y}+\frac{\partial F_r}{\partial z}\frac{\partial \phi_1}{\partial z}, \tag{33}$$

where ν_r is the outward normal to the curve $f_r(y,z)=0$.

The problem of finding a solution of the wave equation with zero normal derivatives on a tube of variable cross-section has thus been transformed into the problem of determining a solution with given normal derivatives on a tube of constant cross-section.

It is seen from equation (29) that since $\phi_1=\phi_2$ at $x=0$, $\Phi=0$ at $x=0$; then Φ is a solution of the equation

$$\nabla^2\Phi-\frac{1}{a^2}\frac{\partial^2\Phi}{\partial t^2}=0 \tag{34}$$

neglecting terms of $O(\epsilon^2)$.

A solution of equation (34) is therefore required with initial conditions that Φ, $\partial\Phi/\partial t$ are zero at $t=0$, and boundary conditions Φ zero at $x=0$ and $\partial\Phi/\partial\nu$ given on the walls of the tube. There is also the condition that Φ is zero everywhere for $t<0$.

The application of Heaviside's operational methods to equation (34) gives

$$\nabla^2\bar{\Phi}-\frac{p^2}{a^2}\bar{\Phi} = 0. \tag{35}$$

At points inside the curve C, a solution of (35) may be expanded in the form

$$\bar{\Phi} = \sum_n \bar{c}_n(x,p)\psi_n(y,z), \tag{36}$$

where $$\bar{c}_n(x,p) = \int_A \bar{\Phi}\,\psi_n^*(y,z)\,dS.$$

The boundary condition on Φ at $x = 0$ implies that $\bar{c}_n(x,p)$ is zero at $x = 0$.

Multiplying equation (35) by $\psi_n^*(y,z)$ and integrating the resulting equation over the area A, it can be shown, by the use of Green's theorem, that

$$\frac{\partial^2\bar{c}_n}{\partial x^2}-\left(k_n^2+\frac{p^2}{a^2}\right)\bar{c}_n+\oint_c \frac{\partial\bar{\Phi}}{\partial\nu}\psi_n^*\,dS = 0. \tag{37}$$

A solution of this equation will be found by the use of a Green's function. The Green's function $G_n(x,x',p)$ is defined such that

$$\frac{\partial^2 G_n}{\partial x^2}-\lambda_n^2\,G_n = -\delta(x-x'), \tag{38}$$

where $\delta(x-x')$ is the Dirac delta function,

$$G_n \to 0 \quad \text{as} \quad x \to \pm\infty,$$

and $G_n = 0$ at $x = 0$. It is easily shown that

$$G_n = \frac{1}{2\lambda_n}[e^{-\lambda_n|x-x'|}-e^{-\lambda_n|x+x'|}] \quad \text{for } x > 0,\ x' > 0. \tag{39}$$

Consider the function

$$\int_0^\infty \frac{\partial^2 G_n}{\partial(x')^2}\bar{c}_n(x',p)\,dx'-\int_0^\infty \frac{\partial^2\bar{c}_n}{\partial(x')^2}(x',p)G_n\,dx' \quad \text{for } x > 0 \tag{40}$$

which is seen to be zero on integrating by parts. Thus substituting from equation (37) in the expression (40) we have

$$\bar{c}_n(x,p) = \int_0^\infty \oint_c \frac{\partial\bar{\Phi}}{\partial\nu}\psi_n^*\,ds\cdot G_n(x,x',p)\,dx' \quad \text{for } x > 0, \tag{41}$$

and a corresponding solution exists for $x < 0$.

Thus a solution of equation (35) has been obtained as a sum of eigenfunctions, and by inversion of the operational form an expression for Φ can be obtained.

It is seen from equation (41) that the change in the velocity potential

caused by a distortion of part of the boundary curve C is independent of the distortion of the rest of C if terms of $O(\epsilon^2)$ are neglected.

7. The effect of distortions in a rectangular tube

Little advantage is to be gained by proceeding with the analysis for a tube of general cross-section any further, and a particular example will now be given.

A tube which at $x = 0$ has a rectangular cross-section, with sides of length b and d, will be considered. Since the contributions to Φ from each of the four walls have been shown to be independent, a distortion of one wall only need be considered.

Let the equations of the walls of the tube be

$$y = \epsilon F(x,z), \quad y = b, \quad z = 0, \quad \text{and} \quad z = d, \tag{42}$$

where F and its derivatives are bounded.

From equation (33) the boundary conditions on the tube walls are

$$\frac{\partial\overline{\Phi}}{\partial y} = -\frac{\partial^2\overline{\Phi}_1}{\partial y^2}F(x,z)+\frac{\partial F}{\partial x}\frac{\partial\overline{\Phi}_1}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial\overline{\Phi}_1}{\partial z} \quad \text{on } y = 0, \tag{43}$$

$\partial\overline{\Phi}/\partial\nu = 0$ everywhere else on C.

A further simplification is now introduced, that the cross-section is assumed to remain rectangular everywhere. Thus F becomes a function of x only, and the third term on the right-hand side of equation (43) becomes zero. The function F will be taken to be equal to $\sin\alpha x$, where α is such that $\alpha\epsilon$ is $O(\epsilon)$. A more general form of $F(x)$ is considered briefly in section 9.

If, at $x = 0$, ϕ_2 consists of a sum of modes, energy will be continually transferred between the modes as the sound waves move along the tube. The case when, at $x = 0$, ϕ_2 consists of a single mode is given below. If this is regarded as the consideration of a single term of a more general form of ϕ at $x = 0$, then the analysis given can be regarded as the energy lost in one mode neglecting the energy gained by that mode from the other modes. An expression for this neglected energy can easily be obtained, however, by a summation over all the modes of the energy converted into the mode under consideration.

Let
$$\bar{\phi} = \bar{a}_{rs}(p)\psi_{rs}(y,z) \quad \text{at } x = 0, \tag{44}$$

where
$$\psi_{rs} = \frac{2}{(bd)^{\frac{1}{2}}}\cos\frac{r\pi y}{b}\cos\frac{s\pi z}{d}\rho_r\rho_s,$$

where
$$\rho_r = 1 \quad (r \neq 0), \qquad \rho_0 = 2^{-\frac{1}{2}},$$

and
$$k_{rs}^2 = \frac{r^2\pi^2}{b^2}+\frac{s^2\pi^2}{d^2}.$$

For a rectangular tube the summation parameter n of the tube of general

section is replaced by the two parameters r and s and a double summation is required, and thus for a rectangular tube equation (36) becomes

$$\bar{\Phi} = \sum_{r,s} \bar{c}_{rs}(x,p)\psi_{rs}(y,z).$$

It has been shown that for $x > 0$ with the value of $\bar{\phi}$ at $x = 0$ given by equation (44) we have

$$\bar{\phi}_1 = \bar{a}_{rs}(p)\psi_{rs}(y,z)e^{-\lambda_{rs}x}, \tag{45}$$

where

$$\lambda_{rs}^2 = k_{rs}^2 + \frac{p^2}{a^2}.$$

Substituting from equation (45) in equation (43), the boundary condition on $y = 0$ can be written

$$\left(\frac{\partial\bar{\Phi}}{\partial y}\right)_{y=0} = [\psi_{rs}(y,z)]_{y=0}\, e^{-\lambda_{rs}x}\bar{a}_{rs}(p)\left[\left(\frac{r\pi}{b}\right)^2 \sin\alpha x - \alpha\cos\alpha x\,\lambda_{rs}\right]. \tag{46}$$

It is easily shown that

$$\left[\int_0^d \frac{\partial\bar{\Phi}}{\partial y}\psi_{nm}^*\,dz\right]_{y=0} = \rho_r\,\rho_n\,\delta_{sm}\frac{2}{b}e^{-\lambda_{rs}x}\bar{a}_{rs}(p)\left[\left(\frac{r\pi}{b}\right)^2\sin\alpha x - \alpha\cos\alpha x\,\lambda_{rs}\right], \tag{47}$$

where δ_{sm} is the Kronecker delta function.

Substituting from (47) in the general expression (41) and noting that

$$\frac{\partial\bar{\Phi}}{\partial y} = -\frac{\partial\bar{\Phi}}{\partial\nu}$$

at $y = 0$, it is seen that

$$\bar{c}_{nm}(x,p) = \delta_{sm}\,\rho_r\,\rho_n\int_0^\infty -\frac{2}{b}\bar{a}_{rs}\,e^{-\lambda_{rs}x'}\left[\sin\alpha x'\left(\frac{r\pi}{b}\right)^2 - \alpha\cos\alpha x'\lambda_{rs}\right]G_{nm}\,dx' \tag{48}$$

for $x > 0$, where

$$G_{nm}(x,x',p) = \frac{1}{2\lambda_{nm}}[e^{-\lambda_{nm}|x-x'|} - e^{-\lambda_{nm}|x+x'|}].$$

In order to carry out the integration in equation (48) the values of two integrals will now be given. It can be shown that

$$\int_0^\infty G_{nm}\,e^{-\lambda_{rs}x'}\sin\alpha x'\,dx'$$

$$= \frac{1}{2}\left[\frac{\lambda_{rs}\,a^2}{\alpha(p^2+\gamma_{nm}^2)}(e^{-\lambda_{nm}x} - e^{-\lambda_{rs}x}\cos\alpha x) + \frac{(\alpha^2+k_{nm}^2-k_{rs}^2)a^2}{2\alpha^2(p^2+\gamma_{nm}^2)}\sin\alpha x\,e^{-\lambda_{rs}x}\right], \tag{49}$$

and

$$\int_0^\infty G_{nm}\, e^{-\lambda_{rs}x'} \cos \alpha x'\, dx'$$

$$= +\frac{1}{2}\left[\frac{\lambda_{rs}a^2}{\alpha(p^2+\gamma_{nm}^2)} \sin \alpha x\, e^{-\lambda_{rs}x} - \frac{(\alpha^2+k_{nm}^2-k_{rs}^2)a^2}{2\alpha^2(p^2+\gamma_{nm}^2)}(e^{-\lambda_{nm}x}-\cos \alpha x\, e^{-\lambda_{rs}x})\right], \tag{50}$$

where γ_{nm} is defined by the equation

$$\gamma_{nm}^2 = a^2\left[\frac{\alpha^2}{4}+\frac{k_{rs}^2+k_{nm}^2}{2}+\frac{(k_{nm}^2-k_{rs}^2)^2}{4\alpha^2}\right].$$

It should be noted that

$$\frac{\gamma_{nm}^2}{a^2}-k_{nm}^2 = \left(\frac{1}{2\alpha}(\alpha^2+k_{rs}^2-k_{nm}^2)\right)^2 \geqslant 0 \tag{51}$$

and

$$\frac{\gamma_{nm}^2}{a^2}-k_{rs}^2 = \left(\frac{1}{2\alpha}(\alpha^2+k_{nm}^2-k_{rs}^2)\right)^2 \geqslant 0. \tag{52}$$

When square roots of the expressions (51), (52) are taken it is understood that the positive root is to be taken.

Substituting from equation (49) and (50) in equation (48) it is seen that

$$\bar{c}_{nm}(x,p) = -\frac{\delta_{sm}\,\rho_r\,\rho_n}{p^2+\gamma_{nm}^2}\frac{a^2}{b}a_{rs}(p)\left\{\left[\left[\left(\frac{r\pi}{b}\right)^2\frac{\alpha^2+k_{nm}^2-k_{rs}^2}{2\alpha^2}-\lambda_{rs}^2\right]\sin \alpha x\, e^{-\lambda_{rs}x}+\right.\right.$$

$$\left.\left.+(e^{-\lambda_{nm}x}-\cos \alpha x\, e^{-\lambda_{rs}x})\left[\frac{\lambda_{rs}}{\alpha}\left(\frac{r\pi}{b}\right)^2+\frac{\lambda_{rs}}{2\alpha}(\alpha^2+k_{nm}^2-k_{rs}^2)\right]\right]\right\}. \tag{53}$$

The operational form of the solution with a general time dependence at $x = 0$ has thus been obtained. A particular form of this dependence will now be considered.

Let $(\phi)_{x=0} = \cos \omega t\, \psi_{rs}(y,z)$ for $t \geqslant 0$, where $k_{r+1,s} > \omega/a > k_{r,s}$.

In the analysis given below, only the steady state or main signal terms of Φ will be considered, the transient terms being neglected to simplify somewhat the form of the results. The region in which the transient terms may be neglected in comparison with the main signal terms will be discussed later. For values of Φ outside this region the transient terms must be included and can be calculated by methods similar to those used in sections 3–5.

The functions p_{nm}, q_{rs}, and ϵ_{rn} are defined by the equations

$$p_{nm} = \left(1-\frac{a^2k_{nm}^2}{\omega^2}\right)^{-\frac{1}{2}}, \qquad q_{nm} = \left(1-\frac{a^2k_{rs}^2}{\gamma_{nm}^2}\right)^{-\frac{1}{2}}$$

and

$$\epsilon_{r,n} = 0 \quad \text{if } n > r, \qquad \epsilon_{r,n} = 1 \quad \text{if } n \leqslant r.$$

Neglecting the transient terms it is easily shown from equation (53) that

$$c_{nm}(x) = \delta_{sm}\,\rho_r\,\rho_n \frac{a^2}{2b\alpha}\frac{1}{\gamma_{nm}^2-\omega^2}\left\{\epsilon_{r,n}\,H\left(t-p_{nm}\frac{x}{a}\right)\times\right.$$

$$\times\left[\alpha^2+k_{nm}^2-k_{rs}^2+2\left(\frac{r\pi}{b}\right)^2\right]\left(\frac{\omega^2}{a^2}-k_{rs}^2\right)^{\frac{1}{2}}\sin\left(\omega t-\frac{x}{a}(\omega^2-k_{nm}^2a^2)^{\frac{1}{2}}\right)+$$

$$+H\left(t-p_{rs}\frac{x}{a}\right)\left[-\left\{\alpha^2+k_{nm}^2-k_{rs}^2+2\left(\frac{r\pi}{b}\right)^2\right\}\right]\left(\frac{\omega^2}{a^2}-k_{rs}^2\right)^{\frac{1}{2}}\cos\alpha x\times$$

$$\times\sin\left(\omega t-\frac{x}{a}(\omega^2-k_{rs}^2a^2)^{\frac{1}{2}}\right)+$$

$$+\left[2k_{rs}^2\alpha-\frac{\alpha^2+k_{nm}^2-k_{rs}^2}{\alpha}\left(\frac{r\pi}{b}\right)^2-\frac{2\omega^2}{a^2}\alpha\right]\sin\alpha x\cos\left(\omega t-\frac{x}{a}(\omega^2-k_{rs}^2a^2)^{\frac{1}{2}}\right)+$$

$$+H\left(t-q_{nm}\frac{x}{a}\right)\left[-\left\{\alpha^2+k_{nm}^2-k_{rs}^2+2\left(\frac{r\pi}{b}\right)^2\right\}\right]\times$$

$$\times\left(\frac{\alpha^2+k_{nm}^2-k_{rs}^2}{2\alpha}\right)\sin\left(\gamma_{nm}t-\frac{x}{a}(\gamma_{mn}^2-k_{nm}^2a^2)^{\frac{1}{2}}\right)+$$

$$+H\left(t-q_{rs}\frac{x}{a}\right)\left[\alpha^2+k_{nm}^2-k_{rs}^2+2\left(\frac{r\pi}{b}\right)^2\right]\frac{(\alpha^2+k_{nm}^2-k_{rs}^2)}{2\alpha}\times$$

$$\left.\times\sin\left(\gamma_{nm}t-\frac{x}{a}(\gamma_{nm}^2-k_{rs}^2a^2)^{\frac{1}{2}}+\alpha x\right)\right\}. \tag{54}$$

In this expression for $c_{nm}(x)$ the transient terms have been neglected. These terms are small compared with the given expression, for values of t such that

$$t \gg p_{rs}x/a, \qquad t \gg p_{ns}x/a \quad \text{for } n \leqslant r, \qquad t \gg q_{rs}x/a, \qquad t \gg q_{nm}x/a.$$

Now $p_{ns} < p_{rs}$ if $n < r$; hence the second condition is implied by the first. Thus if a value of the time $T(\alpha,x,n)$ can be found such that

$$T(\alpha,x,n) = \text{maximum value of } [p_{rs}, q_{rs}, q_{nm}]x/a,$$

then for $t \gg T(\alpha,x,n)$ the transient terms may be neglected.

The function q_{nm} becomes infinite when

$$\alpha^2+k_{rs}^2-k_{nm}^2 = 0 \quad (\alpha \neq 0),$$

and the function q_{rs} becomes infinite when

$$\alpha^2+k_{nm}^2-k_{rs}^2 = 0.$$

Thus for $\alpha^2 \pm (k_{rs}^2-k_{nm}^2) = 0$, T becomes infinite and the third and fourth terms in expression (54) become zero. The transient terms in this case cannot be neglected for any value of the time t.

In the work that follows t will always be taken greater than $T(\alpha,n)$ unless otherwise stated.

From equation (29) it is seen that ϕ_2 has been regarded as the sum of ϕ_1

which has been calculated in sections 2–5 and the term $\epsilon\Phi$ which has been calculated above; $\epsilon\Phi$ represents a disturbance which will be called the secondary wave and which may be regarded as having its origin in the wave ϕ_1 acting on the distortions of the tube walls.

It is seen from the form of equation (54) that $c_{nm}(x)$ is composed of four distinct terms. Each term represents a wave with a velocity which is different in general from the velocities of the waves represented by the other three terms.

The wave represented by the first term vanishes except for modes below cut off and travels with the group velocity of the nsth mode. It can be regarded as representing the accumulation of the secondary waves below cut off, on passing down the tube.

The coefficient of the term in the rsth mode of Φ, i.e. the coefficient of the term in the original mode of ϕ_1, is seen to be $c_{rs}(x)$ from equation (54), and equals

$$(\rho_r)^2\left\{\frac{a^2}{2b\alpha}\frac{\{\alpha^2+2(r\pi/b)^2\}}{\gamma_{rs}^2-\omega^2}\left(\frac{\omega^2}{a^2}-k_{rs}^2\right)^{\frac{1}{2}}(1-\cos\alpha x)\sin\left[\omega t-\left(\frac{\omega^2}{a^2}-k_{rs}^2\right)^{\frac{1}{2}}x\right]+\right.$$

$$+\frac{a^2}{b}\frac{\{k_{rs}^2-\frac{1}{2}(r\pi/b)^2-\omega^2/a^2\}}{(\gamma_{rs}^2-\omega^2)}\sin\alpha x\cos\left[\omega t-\left(\frac{\omega^2}{a^2}-k_{rs}^2\right)^{\frac{1}{2}}x\right]+$$

$$\left.+\frac{a^2}{4b}\frac{\{\alpha^2+2(r\pi/b)^2\}}{(\gamma_{rs}^2-\omega^2)}\{\sin(\gamma_{rs}t+\tfrac{1}{2}\alpha x)-\sin(\gamma_{rs}t-\tfrac{1}{2}\alpha x)\}\right\}.$$

If a pulse of finite duration is considered such that

$$(\phi)_{x=0}=\cos\omega t\{H(t)-H(t-\tau)\}\psi_{rs},$$

then for points such that $t \geqslant q_{rs}x/a+\tau$, i.e. for points behind the main signal,

$$c_{rs}(x)=\frac{a^2}{4b}(\rho_r)^2\frac{\{a^2+2(r\pi/b)^2\}}{(\gamma_{rs}^2-\omega^2)}\left\{\begin{matrix}2\cos(\gamma_{rs}t-\frac{1}{2}\gamma_{rs}\tau+\frac{1}{2}\alpha x)\\-2\cos(\gamma_{rs}t-\frac{1}{2}\gamma_{rs}\tau-\frac{1}{2}\alpha x)\end{matrix}\right\}\sin\frac{\gamma_{rs}\tau}{2}.$$

Thus behind the main signal there are two waves of equal amplitude travelling in opposite directions. Alternatively, the disturbance behind the main signal can be regarded as a system of standing waves.

8. Resonance

It is seen from equation (54) that a resonance phenomenon occurs in the nth mode when $\omega^2=\gamma_{nm}^2$, i.e. when

$$\frac{\omega^2}{a^2}=\frac{\alpha^2}{4}+\frac{k_{rs}^2+k_{nm}^2}{2}+\frac{(k_{nm}^2-k_{rs}^2)^2}{4\alpha^2}.$$

For this value of ω the expression (54) for $c_{nm}(x)$ becomes infinite and is meaningless.

Resonance occurs in the working mode, i.e. the rth, when

$$\frac{\omega^2}{a^2} = \frac{\alpha^2}{4} + k_{rs}^2.$$

A graph of the resonant wave-length of corrugation $(2\pi/\alpha)$ of the tube against the wave-length of the initial disturbance $(2\pi a/\omega)$ is given in Fig. 3. The initial disturbance is taken to be in the 1, 0 mode and graphs of the

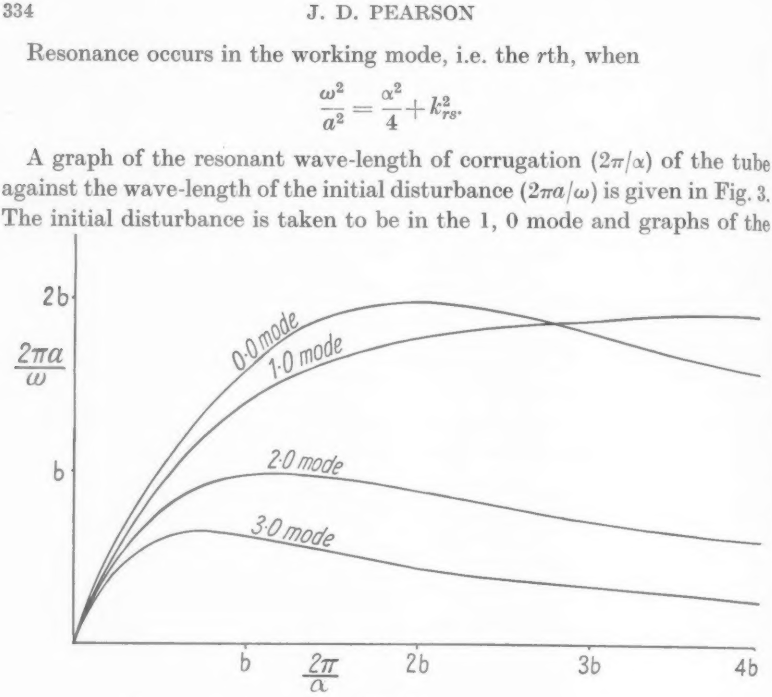

FIG. 3.

resonant wave-lengths for the first four modes are given. It is seen that resonance cannot occur when the wave-length of the disturbance is greater than the critical wave-length of the 1, 0 mode.

It can be shown that, neglecting transient terms,

$$\frac{p^2}{(p^2+\omega^2)^2} e^{-\lambda_{rs} x} \doteqdot \left(t - \frac{\omega x}{a(\omega^2 - k_{rs}^2 a^2)^{\frac{1}{2}}}\right) \frac{1}{2\omega} \sin\left(\omega t - \frac{x}{a}(\omega^2 - k_{rs}^2 a^2)^{\frac{1}{2}}\right),$$

$$\frac{p^4}{(p^2+\omega^2)^2} e^{-\lambda_{rs} x} \doteqdot -\frac{\omega}{2}\left(t - \frac{\omega x}{a(\omega^2 - k_{rs}^2 a^2)^{\frac{1}{2}}}\right) \sin\left(\omega t - \frac{x}{a}(\omega^2 - k_{rs}^2 a^2)^{\frac{1}{2}}\right) + \\ + \cos\left(\omega t - \frac{x}{a}(\omega^2 - k_{rs}^2 a^2)^{\frac{1}{2}}\right),$$

$$\frac{p^2}{(p^2+\omega^2)^2} \lambda_{rs} e^{-\lambda_{rs} x} \\ \doteqdot \frac{1}{2\omega}\left(\frac{\omega^2}{a^2} - k_{rs}^2\right)^{\frac{1}{2}} \left(t - \frac{\omega x}{a(\omega^2 - k_{rs}^2 a^2)^{\frac{1}{2}}}\right) \cos\left(\omega t - \frac{x}{a}(\omega^2 - k_{rs}^2 a^2)^{\frac{1}{2}}\right) + \\ + \frac{1}{2a^2}\left(\frac{\omega^2}{a^2} - k_{rs}^2\right)^{-\frac{1}{2}} \sin\left(\omega t - \frac{x}{a}(\omega^2 - k_{rs}^2 a^2)^{\frac{1}{2}}\right). \quad (55)$$

On substituting these values in equation (53) it is seen that the disturbance

will remain small in the rth mode for values of t such that

$$t \approx \frac{\omega x}{a}(\omega^2 - k_{rs}^2 a^2)^{-\frac{1}{2}},$$

which is satisfied at the front of the main signal.

To obtain an expression for Φ valid at the front of the main signal in the case of resonance, equation (53) must be interpreted by use of equation (55), to the right-hand side of which must be added the transient terms which can be calculated as in section 5.

The solution for Φ so obtained will become invalid at some distance behind the main signal front and will then be comparable with the term ϕ_1. For a short signal, however, Φ will never be comparable with ϕ_1 and thus the solution indicated above will be valid everywhere.

9. More general variation in section of a rectangular tube

If the function $F(x, z)$ of equation (42) is taken to be a function of x only and of the form

$$F(x) = \sqrt{(2/\pi)} \int_0^\infty \sin \alpha x f(\alpha)\, d\alpha \tag{56}$$

then

$$f(\alpha) = \sqrt{(2/\pi)} \int_0^\infty \sin \alpha x\, F(x)\, dx. \tag{57}$$

Now $F(x)$ and $\partial F/\partial x$ must be bounded, and thus the following must be true:

$$\int_0^\infty \sin \alpha x f(\alpha)\, d\alpha = O(1)$$

and

$$\int_0^\infty \alpha \sin \alpha x f(\alpha)\, d\alpha = O(1).$$

These conditions are satisfied if $f(\alpha)$ is of bounded variation and $|\alpha f(\alpha)|$ integrable from 0 to ∞ (**3**, p. 425).

In order that $F(x)$ shall be expressible in the form (56) it must be odd about $x = 0$. The results of sections 6–8 may be extended to this more general form of variation in section by multiplying them by $\sqrt{(2/\pi)} f(\alpha)$ and integrating with respect to α from 0 to ∞. The form (56) could have been taken as the variation in section throughout the analysis of sections 7 and 8, all differentiations, integrations, summations, etc., being then carried on under the integral sign of α. The form of $f(\alpha)$ is assumed to be such that the change in the order of integration and differentiation, etc., is valid.

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A NOTE ON GRAVITY WAVES OF FINITE AMPLITUDE

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SUMMARY

Some corrections are found necessary in Struik's paper on travelling waves. These do not invalidate the proof of the existence of finite travelling waves on water of finite depth, but lead to corrected forms of some of the approximate expressions for the wave profile and the speed of propagation.

It is pointed out that the amplitude of highest waves must decrease with decreasing depth. The definition of the velocity of propagation is considered, and an expression for the velocity to the third order is obtained, which indicates some necessary corrections to certain technical reports.

1. Introduction

THE purpose of this note is to draw attention to, and to correct, some inaccuracies in Struik's paper (**1**) on travelling waves in water of finite depth, and also in some technical reports (**2**, **3**) which use the results obtained in Struik's paper. The corrections are mainly in the third-order terms of the wave profile equation and in the expression for the velocity of propagation. A small correction is also required in the expression given by Stokes (**4**) for the velocity of propagation of these waves. A number of mistakes in sign also occur in that part of Struik's paper which establishes the actual existence of waves of finite height in water of finite depth. For the sake of completeness the necessary corrections are indicated and are found not to invalidate the proof.

Levi-Civita finally established in 1925 the existence of gravity waves of finite height in water of *infinite* depth (**5**). Struik's investigation uses an extension of Levi-Civita's method, but it was recently pointed out by Bowden (**6**) that Struik's solution does not reduce to that of Levi-Civita, nor exactly to the solution obtained by Stokes, when the depth of water tends to infinity. When, however, the above corrections are made, it is found that all the solutions are consistent.

2. Corrections to Struik's proof of the existence of travelling waves in water of finite depth

In Struik's induction proof of the general form of the solution, several errors in sign occur, principally in equations (42), (54), (60)–(63) of Part I of his paper and in section 10, which deals with the convergence of the

method. It will be convenient to summarize briefly those parts of Struik's paper which need altering; the equations in sections 2–4 will be numbered as in his paper, those equations which are different from his being indicated by a dash.

Following Levi-Civita and Struik, cartesian axes are taken with the axis of y vertically upwards with $\phi = \psi = 0$ at the origin and $\psi = q$ on the free surface, ϕ and ψ being the velocity potential and stream function respectively. The complex velocity is $w = u-iv$. The region $0 \leqslant \psi \leqslant q$ is transformed into the annulus $1 \leqslant |\zeta| \leqslant R$ of the ζ-plane, where $\zeta = e^{2\pi i f/c\lambda}$, $R = e^{2\pi q/c\lambda} = e^{\alpha}$, and $f = \phi+i\psi$; λ is the wave-length and c the wave velocity.

Levi-Civita (7) has shown that the free surface condition leads to the mixed equation

$$\zeta \frac{d}{d\zeta}\{w(e^{-\alpha}\zeta)w(e^{\alpha}\zeta)\} - \frac{g\lambda c}{2\pi}\left\{\frac{1}{w(e^{-\alpha}\zeta)} - \frac{1}{w(e^{\alpha}\zeta)}\right\} = 0 \tag{E}$$

to be solved on either boundary $|\zeta| = R$ or $|\zeta| = R^{-1}$ of the annulus Γ, and where w is a holomorphic function of ζ in Γ, which is real on $|\zeta| = 1$.

Struik's transformation of this equation is obtained by writing

$$w = ce^{-m-i\omega}, \qquad \omega = \theta+i\tau,$$

and on $|\zeta| = R$ (i.e. $\zeta = Re^{i\sigma}$) equation (E) becomes

$$\frac{d\tau}{d\sigma} - ke^{-3\tau}\sin\theta = 0, \tag{F}$$

where
$$k = \frac{g\lambda}{2\pi c^2}e^{3m}.$$

His equation (42) should then read

$$\frac{d\tau}{d\sigma} - k_0\theta = kP(\theta,\tau)+x\theta, \tag{42$'$}$$

where
$$x = k-k_0 = \sum_{n=1}^{\infty} k_n\mu^n \tag{43}$$

and
$$P(\theta,\tau) = e^{-3\tau}\sin\theta-\theta. \tag{44}$$

Equation (42′) differs from Struik's equation in the sign of the last term.

To solve equation (42′) the following expansions are required:

$$\omega = \sum_{n=1}^{\infty}\omega_n\mu^n, \qquad \omega_n = \theta_n+i\tau_n,$$

$$P(\theta,\tau) = \sum_{n=1}^{\infty}P_n\mu^n, \qquad kP(\theta,\tau) = \sum_{n=1}^{\infty}\varpi_n\mu^n.$$

The first-order solution is simply $\omega_1 = -i(\zeta+\zeta^{-1})$, $k_0 = \coth\alpha$. Equating powers of μ in equation (42′), Struik's equation (54) takes the form

$$\frac{d\tau_n}{d\sigma} - k_0\theta_n = \chi_n + k_{n-1}\theta_1 \quad (n = 2, 3,...), \tag{54′}$$

where

$$\chi_n = \varpi_n + \sum_{\nu=1}^{n-2} k_\nu \theta_{n-\nu}. \tag{55′}$$

Considering the form of the function $P(\theta, \tau)$ and the assumed form of the functions ω_r $(r = 1, 2,..., n-1)$ and k_r $(r = 1, 2,..., n-2)$, Struik obtains the polynomial for χ_n,

$$\chi_n = q_{n,n}\sin n\sigma + q_{n,n-2}\sin(n-2)\sigma + ... + q_{n,e}\sin e\sigma, \tag{56}$$

where $e = 1$ for n odd and $e = 2$ for n even, the q_{ij}'s being constants. The unknown function ω_n must possess an expansion of the form

$$-i\sum_{\nu=2}^{\infty}\gamma_{n,\nu}\eta_\nu - \sum_{\nu=1}^{\infty}\gamma'_{n,\nu}\delta_\nu,$$

and equation (57) should read

$$\theta_n = -\sum_{\nu=2}^{\infty}\{-D_\nu\gamma_{n,\nu}\sin\nu\sigma + D_\nu\gamma'_{n,\nu}\cos\nu\sigma\} - D_1\gamma'_{n,1}\cos\sigma,$$

$$\tau_n = -\sum_{\nu=2}^{\infty}\{S_\nu\gamma_{n,\nu}\cos\nu\sigma + S_\nu\gamma'_{n,\nu}\sin\nu\sigma\} - S_1\gamma'_{n,1}\sin\sigma, \tag{57′}$$

where $\eta_\nu = \zeta^\nu + \zeta^{-\nu}$, $\delta_\nu = \zeta^\nu - \zeta^{-\nu}$, $S_\nu = R^\nu + R^{-\nu}$, and $D_\nu = R^\nu - R^{-\nu}$.

Substituting equations (57′) in (54′) and equating harmonics in σ, we have the following results which contain differences in sign as compared with equations (60)–(63):

$$\gamma'_{n,\nu} \equiv 0; \qquad \gamma_{n,\nu} = 0 \quad \text{for } \nu > n, \text{ and for } \nu < n \text{ when } n-\nu \text{ is odd;}$$

$$(-\nu S_\nu + k_0 D_\nu)\gamma_{n,\nu} = -q_{n,\nu} \quad \text{for } 2 < \nu < n; \tag{60′}$$

$$q_{n,1} + D_1 k_{n-1} = 0 \quad \text{for } n \text{ odd}, \qquad k_{n-1} = 0 \quad \text{for } n \text{ even}.$$

The required solution is thus given by

$$\omega_n = i\sum_{\nu=e}^{n}{}' \frac{q_{n,\nu}}{-\nu S_\nu + k_0 D_\nu}\eta_\nu, \tag{61′}$$

and

$$k_{n-1} = -\frac{1}{\pi D_1}\int_{-\pi}^{\pi}\chi_n(\sigma)\sin\sigma\,d\sigma = -\frac{1}{\pi D_1}\int_{-\pi}^{\pi}\varpi_n(\sigma)\sin\sigma\,d\sigma, \tag{62′), (63′}$$

where $\sum'$ denotes summation over $\eta_n, \eta_{n-2},..., \eta_e$.

A number of changes of sign are necessary in Struik's section 10 following the changes in equations (42) and (61), but they do not affect the proof of convergence.

3. The amended solution to the third order

To obtain the velocity of propagation and the form of the wave profile to the third order, it is necessary to calculate γ_{22}, γ_{33}, and k_2. It is found that a correction is needed to the value of k_2 obtained by Struik, and some further corrections are needed in the subsequent calculations. The corrected value of k_2 is

$$k_2 = -\frac{S_5+6S_3+11S_1}{D_1^3}. \qquad \text{II}(9')$$

Struik obtains to the third order the correct results

$$e^{-m} = 1+\mu^2 \qquad \text{II}(10)$$

and

$$z = x+iy = \frac{f}{c}-2b\sin\frac{2\pi f}{c\lambda}+\frac{4\pi}{\lambda}b^2\frac{S_2+1}{D_1^2}\sin\frac{4\pi f}{c\lambda}- -\left(\frac{2\pi}{\lambda}\right)^2 b^3\frac{3S_4+4S_2+4}{D_1^4}\sin\frac{6\pi f}{c\lambda}, \qquad \text{II}(17)$$

where, however, the correct form of b is given by

$$-\frac{2\pi}{\lambda}b = \mu+\frac{S_2+4}{S_2-2}\mu^3. \qquad \text{II}(15')$$

The velocity of propagation is found from the equation

$$\frac{g\lambda}{2\pi c^2} = e^{-3m}k = (1+3\mu^2)(k_0+k_2\mu^2), \qquad \text{II}(18')$$

which, after some simplification, gives

$$c^2 = \frac{g\lambda}{2\pi}\frac{D_1}{S_1}\left(1+\frac{S_4+2S_2+12}{D_1^2}\mu^2\right).$$

The last result differs considerably from Struik's result. It agrees with that obtained originally by Stokes (equation (33) of his paper), provided one obvious small correction to Stokes's work is made, namely the insertion of the factor $(2\pi/\lambda)^2$ in the second-order term.

On the free surface l, $f = \phi+iq$, and separating II(17) into real and imaginary parts, Struik obtains Stokes's parametric equations for the wave profile. When, however, ϕ was eliminated between these equations by Struik, he omitted some third-order terms and his explicit wave profile equation II(30) should read

$$y_l = \frac{q}{c}+\frac{\pi}{\lambda}b^2D_2-bD_1\cos\frac{2\pi x}{\lambda}+\left(\frac{2\pi}{\lambda}\right)b^2\frac{S_1(S_2+4)}{2D_1}\cos\frac{4\pi x}{\lambda}- -\left(\frac{2\pi}{\lambda}\right)^2 b^3\left\{\frac{9S_6+10S_4-S_2-36}{8D_1^3}\cos\frac{2\pi x}{\lambda}+\left(\frac{3}{8}\right)\frac{S_6+6S_4+15S_2+28}{D_1^3}\cos\frac{6\pi x}{\lambda}\right\}, \qquad \text{II}(30')$$

which differs from equation II(30) in the coefficient of the second harmonic and in both third-order terms. Struik's expressions for the elevation above and depression below mean level and total-wave amplitude remain unchanged.

If we now write the coefficient of $\cos(2\pi x/\lambda)$ in II(30′) in the form

$$b_1 D_1 = -bD_1 - \frac{9}{8}\left(\frac{2\pi}{\lambda}\right)^2 b^3 D_1^3 + \dots$$

and assume $b_1 D_1 \to a$ as the depth of water tends to infinity, where a is finite, then the limiting form of II(30′), after a suitable change of axes, is

$$y' = a\cos\frac{2\pi x}{\lambda} + \frac{1}{2}\left(\frac{2\pi}{\lambda}\right)a^2\cos\frac{4\pi x}{\lambda} + \frac{3}{8}\left(\frac{2\pi}{\lambda}\right)^2 a^3\cos\frac{6\pi x}{\lambda},$$

which is essentially the same result as that obtained by Stokes for an infinite depth of water.

The corresponding wave amplitudes for a particular depth of water may be determined from the relations

$$H = -\frac{\lambda}{2\pi}\left\{2\mu D_1 + 3\mu^3\frac{D_7 + 2D_5 + 2D_3 - 5D_1}{D_1^4}\right\}$$

and

$$h = \frac{q}{c} + \frac{\lambda}{4\pi}\mu^2 D_2,$$

where H is the height of the waves from crest to trough, and h is the mean depth of water.

4. Note on the limiting cases of infinite depth and zero depth

By an extension of Levi-Civita's use of 'majoring functions' to the present case, Struik showed that a solution exists for a sufficiently small, but non-zero, range of values of μ. Examining the equations satisfied by these functions, an estimate may be obtained for that range, and it will be seen that it tends to zero both in the case of very shallow water and also for an infinite depth.

Using the following equations of Struik's paper

$$\bar{K}\mu D_1 = 2KG(H) \tag{106}$$

and

$$H = (1+R^2)R\mu + JKG(H) + J\bar{K}H, \tag{108}$$

where

$$\bar{K} = K - k_0 \quad\text{and}\quad G(H) = H^2\mathscr{G}(H), \qquad (95), (111)$$

it follows that $\bar{K} < J^{-1}$ since $H > 0$, and $G(H) \geqslant 3H^2$. Since

$$H > (1+R^2)R\mu,$$

then $G(H) > 3(1+R^2)^2R^2\mu^2$, and using (106),

$$\bar{K}\mu D_1 > 6(1+R^2)^2KR^2\mu^2$$

or

$$\bar{K}\{D_1-6R^2\mu(1+R^2)^2\} > 6k_0 R^2\mu(1+R^2)^2,$$

whence we obtain the sufficient condition $\mu < \mu_0$, say, where

$$\mu_0 = \frac{R^2-1}{6R^3(1+R^2)^2(1+Jk_0)}.$$

It follows that the solution is certainly convergent for values of μ satisfying this inequality. As the depth increases, $R \to \infty$ and $\mu_0 \to 0$, and the corresponding range for μ decreases. This appears at first sight to make the wave height vanish in the limiting case of infinite depth of water, but this is not, however, a *necessary* condition, and more refined inequalities can be found which lead to an upper bound for μ which tends to zero less rapidly than μ_0. This indicates the possible existence of the non-zero limit a referred to in section 3; it is known, of course, that a non-zero value of a does exist from Levi-Civita's original work.

It is perhaps worth noting that Levi-Civita's investigation establishes the existence of permanent travelling waves only for quite small (though finite) amplitudes. In fact an examination of the inequalities used in his demonstration restricts his existence proof to waves of height about 1/200 of the wave-length. Some refinements in his inequalities can be made which bring this ratio up to about 1/98. This falls far short of the height (about 1/7) estimated by Michell (8) and others for the sharp-crested waves, but there appears to be no physical criterion which would indicate a limit to the height below this value.

The case when the depth of water decreases to zero is essentially different. In this case $R \to 1$ and again $\mu_0 \to 0$ and this now requires that the wave height shall tend to zero.

5. The velocity of propagation: corrections to certain reports

It was shown by Stokes that it is possible to define the velocity of propagation of permanent waves in a number of different ways, depending on the method adopted for solving the problem. It is not at once obvious that the various methods so far used in problems of this type do employ the same definition of velocity, but it will now be shown that, to the third order, these results are in agreement.

The first method of solution is that used in the supplement to Stokes's paper (4), where the expansion

$$x = \frac{\phi}{c}+\sum_{n=1}^{\infty} A_n\{e^{(nm/c)(\psi+k)}+e^{-(nm/c)(\psi+k)}\}\sin\frac{nm\phi}{c} \tag{1}$$

is employed, where $m = 2\pi/\lambda$ and $\psi = -k$ on the canal bed. This method is essentially the same as that of Struik who employs the complex variable notation. The resulting expression for the velocity of propagation is given to the third order by

$$c^2 = \frac{g\lambda}{2\pi}\frac{D_1}{S_1}\left\{1+\frac{S_4+2S_2+12}{D_1^4}\,b^2\left(\frac{2\pi}{\lambda}\right)^2\right\}, \tag{2}$$

where $D_n = 2\sinh 2n\pi k/c\lambda$, $S_n = 2\cosh 2n\pi k/c\lambda$, and b is the coefficient of the first harmonic in the wave-profile equation.

The second method is an extension of that of Rayleigh, and uses the expansion

$$\frac{\phi}{c} = x+\sum_{n=1}^{\infty} a_n\{e^{nm(y+h)}+e^{-nm(y+h)}\}\sin nmx, \tag{3}$$

where h is the mean depth of the water. Again proceeding to the third order the velocity is given by

$$c^2 = \frac{g\lambda}{2\pi}\frac{D_1'}{S_1'}\left\{1+\frac{S_4'+16}{D_1'^4}\,b^2\left(\frac{2\pi}{\lambda}\right)^2\right\}, \tag{4}$$

where $$D_n' = 2\sinh\frac{2n\pi h}{\lambda}, \qquad S_n' = 2\cosh\frac{2n\pi h}{\lambda}.$$

To show that equations (2) and (4) are identical, we use the relation obtained by Struik to the third order, namely,

$$h = \frac{k}{c}+\frac{\pi}{\lambda}\,b^2\coth\frac{2\pi k}{c\lambda}, \tag{5}$$

using the above definition of b. Substituting in (4), and using the notation of equation (2) we have

$$\begin{aligned} c^2 &= \frac{g\lambda}{2\pi}\frac{D_1}{S_1}\left[1+\frac{1}{2}\left(\frac{2\pi S_1}{\lambda D_1}\,b\right)^2\right]\left[1+\frac{1}{2}\left(\frac{2\pi b}{\lambda}\right)^2\right]^{-1}\left\{1+\frac{S_4+16}{D_1^4}\left(\frac{2\pi b}{\lambda}\right)^2\right\} \\ &= \frac{g\lambda}{2\pi}\frac{D_1}{S_1}\left\{1+\frac{S_4+2S_2+12}{D_1^4}\left(\frac{2\pi b}{\lambda}\right)^2\right\}. \end{aligned} \tag{2}$$

Exactly the same expression has thus been obtained by each method, and so to the third order, the same definition of velocity applies to both methods. Referring equation (3) to fixed axes, we have

$$\frac{\phi}{c} = \sum_{n=1}^{\infty} a_n\{e^{nm(y+h)}+e^{-nm(y+h)}\}\sin nm(x-ct),$$

and thus at each point of space occupied by the fluid referred to fixed axes, the mean horizontal velocity is zero, where 'mean' denotes the average with respect to time. This satisfies Stokes's first definition of velocity and it may be measured from an ordinate through a fixed point on the canal bed.

It should be noted that the expressions given in two reports of the U.S. Beach Erosion Board (2, 3) are incorrect, since equation (2) is used with the values of D_n and S_n appropriate to equation (4). This is the same as omitting the second-order term in equation (5).

Clearly, equation (4) is simpler and more convenient than (2) since it involves the known h instead of k/c.

In conclusion I wish to thank Professor A. T. Price for help and advice in the course of this work.

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GENERALIZED PLANE STRESS PROBLEMS IN INFINITE ELASTIC STRIPS

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SUMMARY

In this paper we obtain solutions in terms of complex potentials to static problems of generalized plane stress for infinite strips of isotropic elastic material. We solve the following types of problems:

I. Specified stresses (including isolated forces) along both straight boundaries. (Sections (1)–(11).)

II. Specified displacements along both straight boundaries. (Section (12).)

III. Specified displacements along one straight boundary and specified stresses along the other. (Sections (13)–(15).)

IV. Isolated forces in the interior of infinite strips with

(*a*) both boundaries stress-free;

(*b*) both boundaries free from displacements;

(*c*) one boundary free from displacements and the other stress-free. (Sections (16)–(21).)

PROBLEMS OF TYPE I

1. Introduction

FOURIER integral solutions to particular problems of infinite strips with specified boundary stresses have been given by Filon (**1**, **2**) and Howland (**3**) in terms of Airy stress functions. Filon's method has been modified by Hopkins (**4**) to solve problems of a strip with one boundary free from displacements. The present investigation gives rise to fewer difficulties with regard to convergence of integrals and applies to specified boundary displacements as well as to specified stresses. Further, it includes cases in which the boundary stress or displacement functions are not expressible as Fourier integrals and deals with isolated boundary forces without recourse to improper integrals.

2. Notation and fundamental equations

The following investigations are confined to elastic material in equilibrium and in a state of generalized plane stress. If there are no body forces, then it is known (see Stevenson, **5**, **6**) that the stresses and displacements may be expressed in terms of complex functions $\Omega(z)$, $\omega(z)$ of a single complex variable z $(= x+iy)$ by means of the equations

$$2\Theta = 2(\widehat{xx}+\widehat{yy}) = \Omega'(z)+\bar{\Omega}'(\bar{z}), \tag{1}$$

$$-2\bar{\Phi} = -2(\widehat{xx}-\widehat{yy}-2i\widehat{xy}) = \bar{z}\Omega''(z)+\omega''(z), \tag{2}$$

$$8\mu D = 8\mu(u+iv) = \kappa\Omega(z)-z\bar{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z}). \tag{3}$$

The notation is that introduced by Stevenson in the papers cited, primes denoting derivatives, D being the complex displacement, μ a Lamé elastic constant, and $\kappa = 3-4\sigma$, where σ is a modified Poisson's ratio which is defined in terms of the Poisson ratio η by the equation $(1-\sigma)(1+\eta) = 1$. The complex potentials $\Omega(z)$, $\omega(z)$ are analytic functions of z and, therefore, possess derivatives of all finite orders at all interior points of the material. From (1), (2) we see that

$$2(\Theta-\overline{\Phi}) = 4(\widetilde{yy}+i\widetilde{xy}) = \Omega'(z)+\overline{\Omega}'(\bar{z})+\bar{z}\Omega''(z)+\omega''(z). \tag{4}$$

We use the notation

$$\widetilde{yy}_{y=0} = yy^0, \qquad \widetilde{yy}_{y=c_0} = \widetilde{yy}^1, \quad \text{etc.},$$

where c_0 is a constant, and

$$f_T(u) = \int\limits_{-\infty}^{\infty} f(t)e^{-iut}\,dt = \alpha_1(u)+i\alpha_2(u), \tag{5}$$

where α_1 and α_2 are real functions of the real variable u. By the Fourier integral theorem

$$f(x) = \frac{1}{\pi}\int\limits_{0}^{\infty} (\alpha_1\cos xu-\alpha_2\sin xu)\,du. \tag{6}$$

Similarly, for functions $\phi(x)$, $\psi(x)$, $\chi(x)$, we write

$$\phi_T(u) = \epsilon_1+i\epsilon_2, \qquad \psi_T(u) = \sigma_1+i\sigma_2, \qquad \chi_T(u) = \tau_1+i\tau_2. \tag{7}$$

We assume that the functions $f(x)$, $\phi(x)$, $\psi(x)$, $\chi(x)$ satisfy conditions sufficient for the validity of the Fourier integral theorem and that α_1, etc., are bounded for all non-negative values of the parameter u. For brevity we shall write

$$S = \sinh yu, \quad C = \cosh yu, \quad \lambda = uc_0, \quad s = \sinh\lambda, \quad c = \cosh\lambda. \tag{8}$$

3. Specified distributions of stress along both straight boundaries

We solve the problem of the strip subjected to the following boundary stresses:

$$(\widetilde{yy}+i\widetilde{xy})^0 = f(x)+i\phi(x) \tag{9}$$

$$\begin{aligned}(\widetilde{yy}+i\widetilde{yx})^1 &= \psi(x)+i\chi(x)\\ &= \frac{1}{\pi}\int\limits_{0}^{\infty}[(\sigma_1+i\tau_1)\cos xu-(\sigma_2+i\tau_2)\sin xu]\,du,\end{aligned} \tag{10}$$

where f, ϕ, ψ, χ are real functions of x. In a previous paper **(7)** it has been

shown that the conditions in (9) may be satisfied by the sum of the two pairs of potentials

$$\Omega_0(z) = \int I(z)\,dz, \qquad \omega_0(z) = -z\Omega_0(z)+2\int \Omega_0(z)\,dz, \tag{11}$$

$$\Omega_1(z) = \int J(z)\,dz, \qquad \omega_1(z) = -z\Omega_1(z), \tag{12}$$

where $$I(z) = \frac{2}{\pi}\int\limits_0^\infty f_T(u)e^{izu}\,du, \qquad J(z) = -\frac{2i}{\pi}\int\limits_0^\infty \phi_T(u)e^{izu}\,du. \tag{13}$$

For any real integrable functions $\beta_1(u)$, $\beta_2(u)$, $\gamma_1(u)$, $\gamma_2(u)$ of the real variable u we define the integrals

$$H(z) = \frac{i}{\pi}\int\limits_0^\infty [h(u)e^{izu}+\bar{h}(u)e^{-izu}]\,du, \text{ where } h(u) = \beta_1(u)+i\beta_2(u), \tag{14}$$

$$K(z) = \frac{1}{\pi}\int\limits_0^\infty [k(u)e^{izu}+\bar{k}(u)e^{-izu}]\,du, \text{ where } k(u) = \gamma_1(u)+i\gamma_2(u). \tag{15}$$

From the results of paper (7) quoted above we see that, since $H(z)$, $K(z)$ have zero real and imaginary parts respectively when $y = 0$, the following pairs of potentials leave the line $y = 0$ stress-free:

$$\Omega_2(z) = \int H(z)\,dz, \qquad \omega_2(z) = -z\Omega_2(z)+2\int \Omega_2(z)\,dz, \tag{16}$$

$$\Omega_3(z) = \int K(z)\,dz, \qquad \omega_3(z) = -z\Omega_3(z). \tag{17}$$

The suffixes 0, 1, 2, 3 will be attached to all quantities derived from Ω_0, Ω_1, Ω_2, Ω_3 respectively. The solutions obtained below are, in the first instance, purely formal, differentiation with respect to z under integral signs being assumed valid whenever necessary. The validity of the solutions is considered after their completion. Problems are solved for elastic material occupying the region R defined by $-\infty \leqslant x \leqslant \infty$, $0 \leqslant y \leqslant c_0$, by using (11), (12) to obtain specified stresses along the boundary $y = 0$ and choosing $h(u)$, $k(u)$ so that (16), (17) give specified stresses along $y = c_0$. From (4), (11), (12) we obtain

$$(\widetilde{yy}+i\widetilde{xy})_0 = \frac{1}{\pi}\int\limits_0^\infty e^{-yu}\{[(1+yu)\alpha_1+iyu\alpha_2]\cos xu+ \\ +[iyu\alpha_1-(1+yu)\alpha_2]\sin xu\}\,du, \tag{18}$$

$$(\widetilde{yy}+i\widetilde{xy})_1 = \frac{1}{\pi}\int\limits_0^\infty e^{-yu}\{[yu\epsilon_2+(1-yu)i\epsilon_1]\cos xu+ \\ +[yu\epsilon_1-(1-yu)i\epsilon_2]\sin xu\}\,du. \tag{19}$$

From (1), (4), (16), (17) we obtain

$$(\widetilde{yy}+i\widetilde{xy})_2 = \frac{i}{\pi}\int\limits_0^\infty \{[-\beta_1\, Syu+i\beta_2(Cyu-S)]\cos xu+ \\ +[\beta_2\, Syu+i\beta_1(Cyu-S)]\sin xu\}\, du, \tag{20}$$

$$(\widetilde{yy}+i\widetilde{xy})_3 = \frac{1}{\pi}\int\limits_0^\infty \{[-\gamma_1\, Syu+i\gamma_2(Cyu+S)]\cos xu+ \\ +[\gamma_2\, Syu+i\gamma_1(Cyu+S)]\sin xu\}\, du, \tag{21}$$

$$(\widetilde{xx}+\widetilde{yy})_2 = \frac{2}{\pi}\int\limits_0^\infty S(\beta_2\cos xu+\beta_1\sin xu)\, du, \tag{22}$$

$$(\widetilde{xx}+\widetilde{yy})_3 = \frac{2}{\pi}\int\limits_0^\infty C(\gamma_1\cos xu-\gamma_2\sin xu)\, du. \tag{23}$$

Comparison of the integrand in (10) with the sum of the integrands given by (18)–(21) when $y = c_0$ yields the following equations for β_1, β_2, γ_1, γ_2:

$$\begin{aligned} \beta_2(s-\lambda c)-\gamma_1\lambda s+e^{-\lambda}[\alpha_1(1+\lambda)+\epsilon_2\lambda]-\sigma_1 &= 0,\\ \beta_2\lambda s+\gamma_1(s+\lambda c)+e^{-\lambda}[\alpha_1\lambda-\epsilon_2(1-\lambda)]+\tau_2 &= 0,\\ \beta_1(s-\lambda c)+\gamma_2\lambda s+e^{-\lambda}[-\alpha_2(1+\lambda)+\epsilon_1\lambda]+\sigma_2 &= 0,\\ -\beta_1\lambda s+\gamma_2(s+\lambda c)+e^{-\lambda}[\alpha_2\lambda+\epsilon_1(1-\lambda)]-\tau_1 &= 0, \end{aligned} \tag{24}$$

whence

$$\begin{aligned} \beta_1(s^2-\lambda^2) &= \alpha_2(\lambda^2-s^2+\lambda+sc)-\sigma_2(\lambda c+s)-\lambda(\lambda\epsilon_1+s\tau_1),\\ \beta_2(s^2-\lambda^2) &= \alpha_1(s^2-\lambda^2-\lambda-sc)+\sigma_1(\lambda c+s)-\lambda(\lambda\epsilon_2+s\tau_2),\\ \gamma_1(s^2-\lambda^2) &= \epsilon_2(\lambda^2-s^2+cs-\lambda)+\tau_2(\lambda c-s)+\lambda(\lambda\alpha_1-s\sigma_1),\\ \gamma_2(s^2-\lambda^2) &= \epsilon_1(s^2-\lambda^2+\lambda-sc)+\tau_1(s-\lambda c)+\lambda(\lambda\alpha_2-s\sigma_2). \end{aligned} \tag{25}$$

The above solution is valid provided that the I, J, H, K integrals exist and the formal expressions for I', J', H', K' are uniformly convergent with respect to z in the region R. In a previous paper (7) the author has shown that I and J integrals are, in general, satisfactory. The integrals for H' and K' obtained by formal differentiation are uniformly convergent in R if

$$\int\limits_0^\infty e^{\lambda}u(|\beta_1|, |\beta_2|, |\gamma_1|, |\gamma_2|)\, du < \infty. \tag{26}$$

From (25) we see that, at $u = \infty$,

$$\beta_1, \beta_2 = O(\sigma_2\lambda e^{-\lambda}, \tau_1\lambda e^{-\lambda})+O(e^{-2\lambda}), \tag{27}$$

$$\gamma_1, \gamma_2 = O(\sigma_1\lambda e^{-\lambda}, \tau_2\lambda e^{-\lambda})+O(e^{-2\lambda}). \tag{28}$$

Assuming for the moment that all integrands are finite for all non-negative

values of u, we see from (26)–(28) that sufficient conditions for validity of the solution are that, at $u = \infty$,

$$\alpha_1, \alpha_2, \epsilon_1, \epsilon_2 = O(1), \qquad \sigma_1, \sigma_2, \tau_1, \tau_2 = o(u^{-3}). \tag{29}$$

We will now illustrate the above theory by considering three particular cases.

4. The infinite strip with specified, symmetrical, distributions of normal pressure along the straight boundaries

In this case

$$\widetilde{yy}^0 = \widetilde{yy}^1 = f(x), \qquad f(-x) = f(x), \qquad \widetilde{xy}^0 = \widetilde{xy}^1 = 0,$$

where $f(x)$ is any given real function. In this case

$$\alpha_1 = \sigma_1, \qquad \alpha_2 = \sigma_2 = \epsilon_1 = \epsilon_2 = \tau_1 = \tau_2 = 0.$$

If $\alpha_1(0)$ is finite, then the H and K integrands are bounded at $u = 0$. (Compare Filon's treatment (**2**).) The potentials for this problem are

$$\Omega(z) = \sum_{r=0}^{3} \Omega_r(z) = \frac{1}{\pi i}\int_0^\infty [e^{izu}(e^{\lambda}-1)+e^{-izu}(e^{-\lambda}-1)]\frac{\alpha_1(u)\,d\lambda}{\lambda(s+\lambda)}, \tag{30}$$

$$\omega(z) = \sum_{r=0}^{3} \omega_r(z) = -z\Omega(z) - \frac{2}{\pi}\int_0^\infty [e^{izu}(e^{\lambda}+\lambda-1)+$$
$$+e^{-izu}(\lambda+1-e^{-\lambda})-4\lambda]\frac{\alpha_1(u)c_0\,d\lambda}{\lambda^2(s+\lambda)}. \tag{31}$$

Provided that (29) is satisfied, the above potentials give a complete solution to the problem, boundedness of the integrand in (31) at $u = 0$ having been obtained by the introduction of a term independent of z. For Gaussian distributions of normal pressure given by

$$f(x) = -\frac{1}{\delta\pi^{\frac{1}{2}}}e^{-x^2/\delta^2}, \qquad \alpha_1(u) = -e^{-\delta^2u^2/4} \tag{32}$$

the parameter δ may have any non-zero value. As $\delta \to 0$ the boundary stress distributions tend to symmetrical, normal, isolated forces, but the solution cannot then be justified without a revision of the notion of integration.

5. The infinite strip with specified, symmetrical, distributions of shear along the straight boundaries

In this case

$$\tau_2 = -\epsilon_2, \qquad \tau_1 = \epsilon_1 = \alpha_1 = \alpha_2 = \sigma_1 = \sigma_2 = 0.$$

The potentials in this case are

$$\Omega(z) = \frac{1}{\pi i}\int\limits_0^\infty [e^{izu}(e^{\lambda}+1)-e^{-izu}(e^{-\lambda}+1)]\frac{\epsilon_2(u)\,d\lambda}{\lambda(s+\lambda)}, \tag{33}$$

$$\omega(z) = -z\Omega(z)+\frac{2c_0}{\pi}\int\limits_0^\infty (e^{izu}-e^{-izu})\frac{\epsilon_2(u)\,d\lambda}{\lambda(s+\lambda)}. \tag{34}$$

If $\epsilon_2(u) = O(u)$ at $u = 0$, the integrands in (33), (34) are bounded at $u = 0$ and, provided that (29) is satisfied, the solution is valid. When this condition is not satisfied the solution requires modifications which are described later in this paper.

6. Symmetrical, normal, isolated boundary forces

In this section we assume potentials corresponding to isolated boundary forces $\pm iY$ at $(0,0)$, $(0,c_0)$ respectively and use the general method given above to obtain additional potentials which make the remainder of the boundaries stress-free. The potentials

$$\Omega(z) = -\frac{2}{\pi}iY\log z, \qquad \omega(z) = -\frac{2}{\pi}iYz\log z \tag{35}$$

correspond (**5**, **6**) to an isolated force Y in the direction of the positive y-axis at a point O on the boundary of elastic material, O being the origin of rectangular cartesian axes $O(x,y)$. These potentials correspond to a simple radial distribution of stress and, hence, leave all points of the real axis, other than the origin, stress-free. Along the line $y = c_0$, however, they give the stress distributions

$$\widetilde{yy}^1 = \frac{-2Yc_0^3}{\pi(x^2+c_0^2)^2} = -f(x), \qquad \widetilde{xy}^1 = \frac{-2Yc_0^2x}{\pi(x^2+c_0^2)^2} = \phi(x). \tag{36}$$

Similarly, the potentials

$$\begin{aligned}\Omega(z) &= \frac{2iY}{\pi}\log(z-ic_0),\\ \omega(z) &= \frac{2iY}{\pi}(z-ic_0)\log(z-ic_0)-\frac{2c_0Y}{\pi}\log(z-ic_0)\end{aligned} \tag{37}$$

correspond to an isolated force Y in the direction of the negative y-axis at the point $x = 0$ of the boundary $y = c_0$, and along the line $y = 0$ give the stresses

$$\widetilde{yy}^0 = -f(x), \qquad \widetilde{xy}^0 = -\phi(x). \tag{38}$$

Thus, for the solution of the problem of an infinite strip subjected to symmetrical, normal, isolated boundary forces, we require the potentials given in (35), (37), together with potentials which have no singularities inside or on the boundary of R and which satisfy the boundary conditions

$$\widetilde{yy}^0 = \widetilde{yy}^1 = f(x), \qquad \widetilde{xy}^0 = -\widetilde{xy}^1 = \phi(x). \tag{39}$$

The required additional potentials are, clearly, the sum of those obtained in sections 4, 5. In the present case

$$\alpha_1 = Y(\lambda+1)e^{-\lambda}, \qquad \epsilon_2 = -Y\lambda e^{-\lambda}$$

and

$$\Omega(z) = \frac{Y}{\pi i}\int\limits_0^\infty (e^{izu}-e^{-izu-\lambda})(e^{\lambda}-2\lambda-1)\frac{e^{-\lambda}\,d\lambda}{\lambda(s+\lambda)},$$

$$\omega(z) = -z\Omega(z)+\frac{2c_0 Y}{\pi}\int\limits_0^\infty \{e^{izu}[(\lambda+1)(e^{-\lambda}-1)-\lambda(2\lambda+1)e^{-\lambda}]+ \tag{40}$$

$$+e^{-izu-\lambda}[(\lambda+1)(e^{-\lambda}-1)-\lambda]+4\lambda(\lambda+1)e^{-\lambda}\}\frac{d\lambda}{\lambda^2(s+\lambda)}.$$

The potentials in (40) give satisfactory solutions for stresses and displacements.

7. Failure of the above general solution

The formal solution obtained in section 3 to the problem of the elastic strip with specified stresses along the boundaries is unsatisfactory because the expressions for $\beta_1, \beta_2, \gamma_1, \gamma_2$ in (25) are, in general, infinite at $u = 0$ and the integrals defined in (14), (15) are divergent at their lower limits. Sneddon (**8**), who has recently given a formal solution to this problem in terms of Airy stress functions, has derived integrals of a similar type. We discuss only the cases for which

$$\beta_1 = O(u^{-2}), \quad \beta_2 = O(u^{-3}), \quad \gamma_1 = O(u^{-2}), \quad \gamma_2 = O(u^{-1}) \text{ as } u \to 0. \tag{41}$$

Equations (41) are suggested by the following considerations. Consider the inequality

$$\int\limits_{-\infty}^{\infty} [|t^p f(t)|,\ |t^p\phi(t)|,\ |t^p\psi(t)|,\ |t^p\chi(t)|]\,dt < \infty. \tag{42}$$

We shall refer to the above as (42) when $p = 0$, as (42)* when $p = 0, 1$, and as (42)** when $p = 0, 1, 2$. If (42)* is satisfied,

$$\alpha_1, \epsilon_1, \sigma_1, \tau_1 = O(1), \qquad \alpha_2, \epsilon_2, \sigma_2, \tau_2 = O(u) \text{ as } u \to 0. \tag{43}$$

From (25), (43) we can easily show that β_1, etc., satisfy (41). In order to make the general solution satisfactory it is essential that we modify the integrands in (14), (15) without violating the boundary conditions along $y = 0, c_0$, or introducing divergence of the integrals at $u = \infty$.

8. Modification of the complex potentials

Formal operations on the following pairs of 'complex potentials' give $\widetilde{yy} \equiv \widetilde{xy} \equiv 0$, for all functions $g(u)$, $m(u)$, $n(u)$:

$$H_1(z) = \frac{i}{\pi}\int\limits_0^\infty [g(u)+\bar{g}(u)]\,du, \qquad \Omega(z) = \int H_1(z)\,dz,$$

$$\omega(z) = -z\Omega(z)+2\int \Omega(z)\,dz, \tag{44}$$

$$H_2(z) = -\frac{1}{\pi}\int\limits_0^\infty zu[m(u)-\bar{m}(u)]\,du, \qquad \Omega(z) = \int H_2(z)\,dz,$$

$$\omega(z) = -z\Omega(z)+2\int \Omega(z)\,dz, \tag{45}$$

$$K_1(z) = \frac{1}{\pi}\int\limits_0^\infty [n(u)+\bar{n}(u)]\,du, \qquad \Omega(z) = \int K_1(z)\,dz,$$

$$\omega(z) = -z\Omega(z). \tag{46}$$

The potentials in (44) correspond to rigid body displacements, those in (45) to tensions which are constant across the strip, and those in (46) to tensions which increase linearly from zero as we cross the strip from $y = 0$ to $y = c_0$. They may be subtracted from the potentials given by (14), (15) without affecting the stresses along the straight boundaries of the strip. We therefore consider the potentials

$$H^*(z) = H(z)-H_1(z)-H_2(z), \qquad K^*(z) = K(z)-K_1(z), \tag{47}$$

where $g(u) = m(u) = h(u)$, $n(u) = k(u)$. We define potentials Ω_2^*, ω_2^*, Ω_3^*, ω_3^* in terms of H^*, K^* by equations similar to (16), (17). From (47) we obtain

$$H^*(z) = \frac{i}{\pi}\int\limits_0^\infty [(\beta_1+i\beta_2)(e^{izu}-1-izu)+(\beta_1-i\beta_2)(e^{-izu}-1+izu)]\,du, \tag{48}$$

$$K^*(z) = \frac{1}{\pi}\int\limits_0^\infty [(\gamma_1+i\gamma_2)(e^{izu}-1)+(\gamma_1-i\gamma_2)(e^{-izu}-1)]\,du. \tag{49}$$

Assuming that (41) is satisfied, we can easily see that the integrands in (48), (49) are bounded at $u = 0$, whilst, if (27), (28) are satisfied, the

additional terms cannot cause divergence of the integrals at $u = \infty$. Using the modified integrals, the four potentials Ω_0, Ω_1, Ω_2^*, Ω_3^* become

$$\Omega_0(z) = \frac{2}{\pi i}\int\limits_0^\infty (\alpha_1+i\alpha_2)(e^{izu}-1)\frac{du}{u}, \tag{50}$$

$$\Omega_1(z) = -\frac{2}{\pi}\int\limits_0^\infty (\epsilon_1+i\epsilon_2)(e^{izu}-1)\frac{du}{u}, \tag{51}$$

$$\Omega_2^*(z) = \frac{1}{\pi}\int\limits_0^\infty [(\beta_1+i\beta_2)(e^{izu}-1-izu+z^2u^2/2)- -(\beta_1-i\beta_2)(e^{-izu}-1+izu+z^2u^2/2)]\frac{du}{u}, \tag{52}$$

$$\Omega_3^*(z) = \frac{1}{\pi i}\int\limits_0^\infty [(\gamma_1+i\gamma_2)(e^{izu}-1-izu)-(\gamma_1-i\gamma_2)(e^{-izu}-1+izu)]\frac{du}{u}. \tag{53}$$

We have added further rigid body displacement terms to ensure convergence at $u = 0$. Determination of ω_0, ω_2^* requires second integrations and the introduction of additional terms which are independent of z to retain convergence at $u = 0$. The process presents no difficulty but is not described here in detail for the sake of brevity, $\omega(z)$ being required neither for the determination of stresses nor displacements. We have now obtained a complete solution to the problem of the infinite strip with specified stresses along the straight boundaries. This solution is, however, not in a form convenient for the determination of conditions at infinity and we therefore consider alternative ways of modifying the potentials.

9. Alternative method of modification of the potentials

In order to proceed further we assume, in addition to the conditions imposed by (41), that the functions β_1, β_2, γ_1, γ_2 have the following forms:

$$\beta_1 = \frac{A_1}{u^2}+\frac{A_2}{u}+\beta_{1,0}, \tag{54}$$

$$\beta_2 = \frac{B_1}{u^3}+\frac{B_3}{u}+\beta_{2,0}, \tag{55}$$

$$\gamma_1 = \frac{M_1}{u^2}+\gamma_{1,0}, \tag{56}$$

$$\gamma_2 = \frac{N_1}{u}+\gamma_{2,0}, \tag{57}$$

where A_1, A_2, B_1, B_3, M_1, N_1 are constants, and $\beta_{1,0}$, etc., are bounded at $u = 0$; note that β_2 contains no term B_2/u^2 and γ_1 contains no term M_2/u.

Sufficient conditions for the validity of (54)–(57) are that the boundary stress functions satisfy (42)**. However, whereas (54)–(57) cover all physical cases, (42)** are not always satisfied. If we let

$$g(u) = \beta_1(u), \qquad m(u) = iB_1/u^3, \qquad n(u) = M_1/u^2,$$

in (44)–(46), then the integrals thus obtained are convergent at their upper limits and we may use the divergence at their lower limits to cancel that of $H(z)$, $K(z)$. With the above interpretations of $g(u)$, etc., we again consider the modified integrals H^*, K^* defined in (47). We obtain

$$H^*(z) = \frac{i}{\pi}\int\limits_0^\infty [(\beta_1+i\beta_2)e^{izu}+(\beta_1-i\beta_2)e^{-izu}-2\beta_1(u)+2B_1 z/u^2]\,du, \tag{58}$$

$$K^*(z) = \frac{1}{\pi}\int\limits_0^\infty [(\gamma_1+i\gamma_2)e^{izu}+(\gamma_1-i\gamma_2)e^{-izu}-2M_1/u^2]\,du. \tag{59}$$

Assuming (54)–(57), we can show that the integrands in (58), (59) are bounded at $u = 0$. The potentials determined from the above modified integrals are more convenient than those given in (52), etc., for the determination of conditions at infinity. If (55) had contained a term B_2/u^2 or (56) a term M_2/u, then the required modifying integrals would have been divergent at $u = \infty$, and reversion to the previous method of modification is necessary. The potentials Ω_0, Ω_1 given in (50), (51) are required in all cases to give specified conditions along $y = 0$, whilst, using H^*, K^* in (16), (17), the required conditions along $y = c_0$ are given by

$$\Omega_2^*(z) = \frac{1}{\pi}\int\limits_0^\infty \left[(\beta_1+i\beta_2)(e^{izu}-1)-(\beta_1-i\beta_2)(e^{-izu}-1)-2\beta_1\, izu+\frac{iB_1 z^2}{u}\right]\frac{du}{u}, \tag{60}$$

$$\Omega_3^*(z) = \frac{1}{\pi i}\int\limits_0^\infty \left[(\gamma_1+i\gamma_2)(e^{izu}-1)-(\gamma_1-i\gamma_2)(e^{-izu}-1)-\frac{2M_1\, iz}{u}\right]\frac{du}{u}. \tag{61}$$

From (50) we have

$$\int \Omega_0(z)\,dz = -\frac{2}{\pi}\int\limits_0^\infty (\alpha_1+i\alpha_2)(e^{izu}-1-izu)\frac{du}{u^2}, \tag{62}$$

and, from (60),

$$\int \Omega_2^*(z)\,dz = \frac{1}{\pi i}\int\limits_0^\infty [(\beta_1+i\beta_2)(e^{izu}-1-izu)+$$
$$+(\beta_1-i\beta_2)(e^{-izu}-1+izu)+\beta_1 z^2u^2-B_1 z^3/3]\frac{du}{u^2}. \tag{63}$$

The modified complex potentials of the general problem are

$$\left.\begin{aligned}\Omega(z) &= \Omega_0(z)+\Omega_1(z)+\Omega_2^*(z)+\Omega_3^*(z)\\ \omega(z) &= -z\Omega(z)+2\int[\Omega_0(z)+\Omega_2^*(z)]\,dz\end{aligned}\right\}. \tag{64}$$

10. Fundamental integral formulae

The following formulae are required in the subsequent analysis:

(i) If (42) is satisfied, then

$$\lim_{x\to\pm\infty}\int_0^\infty f(u)\begin{Bmatrix}\cos xu\\ \sin xu\end{Bmatrix}du = 0. \tag{65}$$

(ii) If, in addition to satisfying (42), $f(u)$ is of bounded variation in the neighbourhood of $u = 0+$, then

$$\lim_{x\to\pm\infty}\int_0^\infty \frac{f(u)}{u}\sin xu\,du = \pm\tfrac{1}{2}\pi f(0+). \tag{66}$$

(iii)

$$\int_0^\infty \frac{1-\cos xu}{u}\,du = \tfrac{1}{2}\pi|x|. \tag{67}$$

11. Stresses at infinity

Using (65) we can show that, at $x = \pm\infty$,

$$\widetilde{xx}_0 = \widetilde{yy}_0 = \widetilde{xy}_0 = \widetilde{xx}_1 = \widetilde{yy}_1 = \widetilde{xy}_1 = 0. \tag{68}$$

Referring to the potentials Ω_2^*, Ω_3^* derived from H^*, K^* respectively, the equations corresponding to (22), (23) are

$$(\widetilde{xx}+\widetilde{yy})_2 = \frac{2}{\pi}\int_0^\infty [S(\beta_2\cos xu+\beta_1\sin xu)-yB_1/u^2]\,du, \tag{69}$$

$$(\widetilde{xx}+\widetilde{yy})_3 = \frac{2}{\pi}\int_0^\infty [C(\gamma_1\cos xu-\gamma_2\sin xu)-M_1/u^2]\,du. \tag{70}$$

Equations (20), (21) remain unchanged by the modifications in the integrals. Using (20), (21), (69), (70) and the formulae in (65)–(67), we obtain for the stresses at $x = \pm\infty$ respectively

$$\begin{gathered}\widetilde{yy} = 0, \qquad \widetilde{xy} = \pm y(B_1y+2M_1)/2,\\ \widetilde{xx} = \lim_{x\to\pm\infty}[-|x|(B_1y+M_1)\pm(A_1y-N_1)].\end{gathered} \tag{71}$$

In order to proceed further in the general case we assume that (42)** is satisfied. With this assumption we obtain from (5) that

$$f'_T(u) = \alpha'_1(u)+i\alpha'_2(u) = -i\int\limits_{-\infty}^{\infty} tf(t)e^{-iut}\,dt, \tag{72}$$

where the primes denote differentiation with respect to u. Similar expressions may be obtained for $\phi'_T(u)$, $\psi'_T(u)$, $\chi'_T(u)$. Further, it follows that, at $u = 0$,

$$\begin{aligned} \alpha_1,\ \epsilon_1,\ \sigma_1,\ \tau_1,\ \alpha'_2,\ \epsilon'_2,\ \sigma'_2,\ \tau'_2 &= O(1), \\ \alpha'_1,\ \epsilon'_1,\ \sigma'_1,\ \tau'_1,\ \alpha_2,\ \epsilon_2,\ \sigma_2,\ \tau_2 &= O(u). \end{aligned} \tag{73}$$

From (25), (73) we can show that the conditions in the neighbourhood of $u = 0$ are as given in (54)–(57) and that the constants A_1, B_1, M_1, N_1 are given by

$$\begin{aligned} c_0^3 A_1 &= 6[\alpha'_2(0)-\sigma'_2(0)]-3c_0[\epsilon_1(0)+\tau_1(0)], \\ c_0^3 B_1 &= 6[\sigma_1(0)-\alpha_1(0)], \\ c_0^2 M_1 &= 3[\alpha_1(0)-\sigma_1(0)], \\ c_0^2 N_1 &= 3[\alpha'_2(0)-\sigma'_2(0)]-c_0[2\epsilon_1(0)+\tau_1(0)]. \end{aligned} \tag{74}$$

The symbol $\{X, Y, N\}$ will be used to denote a force wrench at the origin which has force resultant $X+iY$ and anti-clockwise moment N, and the suffixes 0, 1 denote the force wrenches at the origin which are statically equivalent to the external stresses applied along the boundaries $y = 0, c_0$ respectively. From (5), (7), (72), etc., we obtain expressions for these wrenches in terms of $\alpha_1(0)$, etc. For example,

$$X_0 = -\int\limits_{-\infty}^{\infty} \widehat{xy}^0\,dx = -\epsilon_1(0).$$

We thus obtain

$$\{X_0, Y_0, N_0\} = \{-\epsilon_1(0), -\alpha_1(0), \alpha'_2(0)\}, \tag{75}$$

$$\{X_1, Y_1, N_1\} = \{\tau_1(0), \sigma_1(0), -\sigma'_2(0)-c_0\tau_1(0)\}. \tag{76}$$

From (71), the force wrenches at the origin which are equivalent to the stresses at $x = \pm\infty$ are given by

$$\lim_{x\to\pm\infty}\{-(X_0+X_1)/2, -(Y_0+Y_1)/2, [x(Y_0+Y_1)-(N_0+N_1)]/2\}. \tag{77}$$

Thus equilibrium of the strip is maintained by the wrenches shown in Fig. 1. It is evident that infinite stresses occur at infinity if and only if $(Y_0+Y_1) \neq 0$. Their existence is due to the fact that, if transverse balancing forces occur at infinity, then their moment about the origin is infinite, and, subject to (42)*, the infinite couples required for equilibrium of the half-strips $x \geqslant 0$,

$x \leqslant 0$ can be obtained only by infinite values of $\widetilde{xx}$. In the diagram the strip is assumed to occupy the region between $x = \pm l$, where l is large but finite.

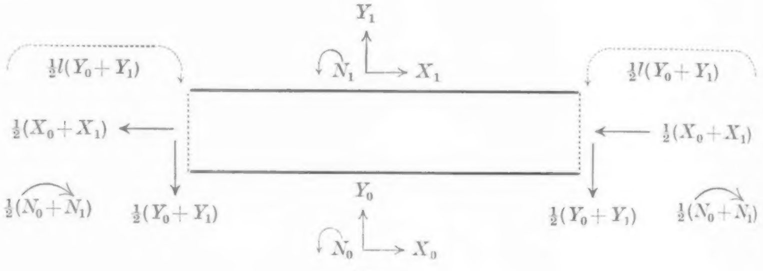

FIG. 1.

PROBLEMS OF TYPE II

12. The infinite strip with specified displacements along the straight boundaries

The author has shown previously (7) that the condition

$$D^0 = f(x)+i\phi(x) \tag{78}$$

may be satisfied by the sum of the two pairs of potentials

$$\Omega_0(z) = \frac{4\mu}{\pi\kappa}\int\limits_0^\infty e^{izu}f_T(u)\,du, \qquad \omega_0(z) = -z\Omega_0(z)+(1-\kappa)\int \Omega_0(z)\,dz, \tag{79}$$

$$\Omega_1(z) = \frac{4\mu i}{\pi\kappa}\int\limits_0^\infty e^{izu}\phi_T(u)\,du, \qquad \omega_1(z) = -z\Omega_1(z)+(1+\kappa)\int \Omega_1(z)\,dz. \tag{80}$$

Also, the potentials

$$\Omega_2(z) = \frac{2\mu}{\kappa}H(z), \qquad \omega_2(z) = -z\Omega_2(z)+(1-\kappa)\int \Omega_2(z)\,dz, \tag{81}$$

$$\Omega_3(z) = \frac{2\mu}{\kappa}K(z), \qquad \omega_3(z) = -z\Omega_3(z)+(1+\kappa)\int \Omega_3(z)\,dz, \tag{82}$$

leave the line $y = 0$ free from displacements. The functions H, K are defined in (14), (15). As in the previous case, the suffixes 0, 1, 2, 3 will be

attached to all quantities derived from Ω_0, Ω_1, Ω_2, Ω_3 respectively. Using (3) we obtain for the displacements along the line $y = c_0$,

$$\begin{aligned}
\pi\kappa D_0^1 &= \int_0^\infty e^{-\lambda}\{[(\kappa-\lambda)\alpha_1+i\lambda\alpha_2]\cos xu+[i\lambda\alpha_1-(\kappa-\lambda)\alpha_2]\sin xu\}\,du,\\
\pi\kappa D_1^1 &= \int_0^\infty e^{-\lambda}\{[i(\kappa+\lambda)\epsilon_1+\lambda\epsilon_2]\cos xu+[\lambda\epsilon_1-i(\kappa+\lambda)\epsilon_2]\sin xu\}\,du,\\
\pi\kappa D_2^1 &= \int_0^\infty \{[-i\beta_1\lambda s+\beta_2(\kappa s+\lambda c)]\cos xu+[\beta_1(\kappa s+\lambda c)+i\beta_2\lambda s]\sin xu\}\,du,\\
\pi\kappa D_3^1 &= \int_0^\infty \{[\gamma_1\lambda s+i\gamma_2(\lambda c-\kappa s)]\cos xu+[i\gamma_1(\lambda c-\kappa s)-\gamma_2\lambda s]\sin xu\}\,du.
\end{aligned} \tag{83}$$

If we require

$$D^1 = \psi(x)+i\chi(x) = \frac{1}{\pi}\int_0^\infty [(\sigma_1+i\tau_1)\cos xu-(\sigma_2+i\tau_2)\sin xu]\,du \tag{84}$$

we have, by comparison of the integrand in (84) with the sum of the integrands in (83),

$$\begin{aligned}
\beta_2(\kappa s+\lambda c)+\gamma_1\lambda s+p &= 0,\\
-\beta_1\lambda s+\gamma_2(\lambda c-\kappa s)+q &= 0,\\
\beta_1(\lambda c+\kappa s)-\gamma_2\lambda s+r &= 0,\\
\beta_2\lambda s+\gamma_1(\lambda c-\kappa s)+t &= 0,
\end{aligned} \tag{85}$$

where

$$\begin{aligned}
p &= e^{-\lambda}[(\kappa-\lambda)\alpha_1+\lambda\epsilon_2]-\kappa\sigma_1,\\
q &= e^{-\lambda}[(\kappa+\lambda)\epsilon_1+\lambda\alpha_2]-\kappa\tau_1,\\
r &= e^{-\lambda}[(\lambda-\kappa)\alpha_2+\lambda\epsilon_1]+\kappa\sigma_2,\\
t &= e^{-\lambda}[-(\kappa+\lambda)\epsilon_2+\lambda\alpha_1]+\kappa\tau_2,
\end{aligned} \tag{86}$$

and, therefore,

$$\begin{aligned}
\beta_1(\kappa^2s^2-\lambda^2) &= r(\lambda c-\kappa s)+q\lambda s,\\
\gamma_2(\kappa^2s^2-\lambda^2) &= q(\lambda c+\kappa s)+r\lambda s,\\
\beta_2(\kappa^2s^2-\lambda^2) &= p(\lambda c-\kappa s)-t\lambda s,\\
\gamma_1(\kappa^2s^2-\lambda^2) &= t(\lambda c+\kappa s)-p\lambda s.
\end{aligned} \tag{87}$$

The integrands in (14), (15) are not always finite at $u = 0$. They are satisfactory if

$$\beta_1, \gamma_1 = O(1), \qquad \beta_2, \gamma_2 = O(u^{-1}) \text{ as } u \to 0. \tag{88}$$

We assume that $\kappa > 1$ and consider the following cases:

(i) $\alpha_1, \alpha_2, \epsilon_1, \epsilon_2, \sigma_1, \sigma_2, \tau_1, \tau_2 = O(1)$,

$$p, q, r, t = O(1), \quad \beta_1, \beta_2, \gamma_1, \gamma_2 = O(u^{-1}) \text{ as } u \to 0. \tag{89}$$

(ii) $\alpha_1, \epsilon_1, \sigma_1, \tau_1 = O(1), \qquad \alpha_2, \epsilon_2, \sigma_2, \tau_2 = O(u)$ as $u \to 0$. (90)

(iii) $\alpha_1, \alpha_2, \epsilon_1, \epsilon_2, \sigma_1, \sigma_2, \tau_1, \tau_2 = O(1)$,

$$\alpha_2-\sigma_2 = O(u), \qquad \epsilon_2-\tau_2 = O(u) \text{ as } u \to 0. \tag{91}$$

When either (ii) or (iii) are satisfied,

$$p, q = O(1), \qquad r, t = O(u), \qquad \beta_1, \gamma_1 = O(1), \qquad \beta_2, \gamma_2 = O(u^{-1}).$$

The conditions given in (89) are the most general of the three, those in (90) occur when the boundary functions satisfy (42), whilst those in (91) occur in important physical cases which we shall consider later in this paper. It can be seen that the conditions in (88) are satisfied in cases (ii), (iii), but that in case (i) the H, K integrals are divergent at the lower limits. We therefore seek potentials which do not violate the conditions along the lines $y = 0$, c_0 but restore convergence, when necessary, at $u = 0$. From (3) it can be seen that, formally, the following pairs of potentials give zero displacements everywhere:

$$\Omega_4(z) = \frac{4\mu i}{\pi\kappa}\int\limits_0^\infty \beta_1(u)\,du, \qquad \omega_4(z) = -z\Omega_4(z)+(1-\kappa)\int \Omega_4(z)\,dz, \tag{92}$$

$$\Omega_5(z) = \frac{4\mu}{\pi\kappa}\int\limits_0^\infty \gamma_1(u)\,du, \qquad \omega_5(z) = -z\Omega_5(z)+(1+\kappa)\int \Omega_5(z)\,dz. \tag{93}$$

Using the above potentials we obtain modified potentials Ω_2^*, ω_2^*, Ω_3^*, ω_3^*, corresponding to Ω_2, ω_2, Ω_3, ω_3 respectively, as follows:

$$\left.\begin{aligned}\Omega_2^*(z) &= \Omega_2(z)-\Omega_4(z) = \frac{2\mu i}{\pi\kappa}\int\limits_0^\infty [(\beta_1+i\beta_2)(e^{izu}-1)+\\ &\qquad\qquad +(\beta_1-i\beta_2)(e^{-izu}-1)]\,du,\\ \omega_2^*(z) &= -z\Omega_2^*(z)+(1-\kappa)\frac{2\mu}{\pi\kappa}\int\limits_0^\infty [(\beta_1+i\beta_2)(e^{izu}-1-izu)-\\ &\qquad\qquad -(\beta_1-i\beta_2)(e^{-izu}-1+izu)]\frac{du}{u},\end{aligned}\right\} \tag{94}$$

$$\left.\begin{aligned}\Omega_3^*(z) &= \Omega_3(z)-\Omega_5(z) = \frac{2\mu}{\pi\kappa}\int\limits_0^\infty [(\gamma_1+i\gamma_2)(e^{izu}-1)+\\ &\qquad\qquad +(\gamma_1-i\gamma_2)(e^{-izu}-1)]\,du,\\ \omega_3^*(z) &= -z\Omega_3^*(z)-(1+\kappa)\frac{2\mu i}{\pi\kappa}\int\limits_0^\infty [(\gamma_1+i\gamma_2)(e^{izu}-1-izu)-\\ &\qquad\qquad -(\gamma_1-i\gamma_2)(e^{-izu}-1+izu)]\frac{du}{u}.\end{aligned}\right\} \tag{95}$$

The above integrals are convergent at their lower limits if (89) is satisfied. Evaluation of stresses as well as displacements from the above potentials requires the determination of Ω'', ω''; the integrals obtained by two formal

differentiations of (81), (82) or (94), (95) are uniformly convergent in R and the solution is satisfactory if

$$\int_0^\infty e^{\lambda} u^2(|\beta_1|, |\beta_2|, |\gamma_1|, |\gamma_2|)\, du < \infty, \tag{96}$$

which requires, from (86), (87), that

$$\beta_1, \beta_2, \gamma_1, \gamma_2 = o(e^{-\lambda} u^{-3}), \qquad p, q, r, t = o(u^{-4}) \text{ as } u \to \infty. \tag{97}$$

Hence the potential integrals and their derivatives are satisfactory at their upper limits if

$$\alpha_1, \alpha_2, \epsilon_1, \epsilon_2 = O(1), \qquad \sigma_1, \sigma_2, \tau_1, \tau_2 = o(u^{-4}) \text{ as } u \to \infty. \tag{98}$$

Problems of Type III

13. Specified displacements along one straight boundary and specified stresses along the other

The potentials given in (79), (80) may be used to obtain specified displacements along the boundary $y = 0$ and the potentials in (81), (82) or (94), (95) may then be chosen to give desired conditions of stress along the boundary $y = c_0$. From (79)–(82), (4) we obtain the following expressions for the stresses given by the above four pairs of potentials along $y = c_0$:

$$(\widetilde{yy}+i\widetilde{xy})_0^1 = \frac{\mu}{\pi\kappa}\int_0^\infty e^{-\lambda}\{[\alpha_2 u(\kappa-1-2\lambda)+i\alpha_1 u(2\lambda-\kappa-1)]\cos xu+$$

$$+[\alpha_1 u(\kappa-1-2\lambda)+i\alpha_2 u(\kappa+1-2\lambda)]\sin xu\}\, du,$$

$$(yy+ixy)_1^1 = \frac{\mu}{\pi\kappa}\int_0^\infty e^{-\lambda}\{[\epsilon_1 u(-1-\kappa-2\lambda)+i\epsilon_2 u(1-\kappa-2\lambda)]\cos xu+$$

$$+[\epsilon_2 u(1+\kappa+2\lambda)+i\epsilon_1 u(1-\kappa-2\lambda)]\sin xu\}\, du,$$

$$(\widetilde{yy}+i\widetilde{xy})_2^1 = \frac{\mu}{\pi\kappa}\int_0^\infty \{[u\beta_1(s-\kappa s-2\lambda c)+iu\beta_2(c+\kappa c+2\lambda s)]\cos xu+$$

$$+[u\beta_2(\kappa s-s+2\lambda c)+iu\beta_1(\kappa c+c+2\lambda s)]\sin xu\}\, du,$$

$$(\widetilde{yy}+i\widetilde{xy})_3^1 = \frac{\mu}{\pi\kappa}\int_0^\infty \{[u\gamma_2(2\lambda s-c-\kappa c)+iu\gamma_1(2\lambda c+s-\kappa s)]\cos xu+$$

$$+[iu\gamma_2(\kappa s-s-2\lambda c)+u\gamma_1(2\lambda s-c-\kappa c)]\sin xu\}\, du. \tag{99}$$

If we require the stresses along $y = c_0$ which are given in (10), we compare the sum of the integrands in (99) with that in (10) and obtain the equations

$$\begin{aligned} \beta_1 u(s-\kappa s-2\lambda c)+\gamma_2 u(2\lambda s-c-\kappa c)+p &= 0, \\ \beta_2 u(c+\kappa c+2\lambda s)+\gamma_1 u(2\lambda c+s-\kappa s)+q &= 0, \\ \beta_2 u(2\lambda c+\kappa s-s)+\gamma_1 u(2\lambda s-c-\kappa c)+r &= 0, \\ \beta_1 u(c+\kappa c+2\lambda s)+\gamma_2 u(\kappa s-s-2\lambda c)+t &= 0, \end{aligned} \tag{100}$$

where

$$\begin{aligned} e^\lambda p &= u\alpha_2(\kappa-1-2\lambda)-u\epsilon_1(\kappa+1+2\lambda)-\kappa e^\lambda \sigma_1/\mu, \\ e^\lambda q &= u\alpha_1(2\lambda-\kappa-1)+u\epsilon_2(1-\kappa-2\lambda)-\kappa e^\lambda \tau_1/\mu, \\ e^\lambda r &= u\alpha_1(\kappa-1-2\lambda)+u\epsilon_2(\kappa+1+2\lambda)+\kappa e^\lambda \sigma_2/\mu, \\ e^\lambda t &= u\alpha_2(\kappa+1-2\lambda)+u\epsilon_1(1-\kappa-2\lambda)+\kappa e^\lambda \tau_2/\mu, \end{aligned} \tag{101}$$

and therefore

$$\begin{aligned} g\beta_1 u &= p(2\lambda c+s-\kappa s)+t(2\lambda s-c-\kappa c), \\ g\gamma_2 u &= p(c+\kappa c+2\lambda s)+t(2\lambda c+\kappa s-s), \\ g\beta_2 u &= q(2\lambda s-c-\kappa c)+r(\kappa s-s-2\lambda c), \\ g\gamma_1 u &= q(s-\kappa s-2\lambda c)+r(c+\kappa c+2\lambda s), \end{aligned} \tag{102}$$

where

$$g = 4\lambda^2+(\kappa+1)^2c^2-(\kappa-1)^2s^2 = 4\lambda^2+\kappa^2+1+2\kappa(c^2+s^2). \tag{103}$$

Using the above expressions for β_1, etc., the required potentials are the sum of those given in (79)–(82). The H, K integrals are satisfactory at $u = 0$ if (88) is satisfied, and this is certainly so if

$$\alpha_1,\ \epsilon_1,\ \sigma_1,\ \tau_1 = O(1), \qquad \alpha_2,\ \epsilon_2,\ \sigma_2,\ \tau_2 = O(u) \text{ as } u \to 0, \tag{104}$$

for in such cases

$$p, q = O(1), \quad r, t = O(u), \quad \beta_1, \gamma_1 = O(u), \quad \beta_2, \gamma_2 = O(1) \text{ as } u \to 0. \tag{105}$$

However, we can show that (89) applies to this problem as well as to the last so that boundedness of $\alpha_1(0)$, etc., is insufficient to ensure that the potential integrals are satisfactory at their lower limits. The modified integrals given in (94), (95) are, however, satisfactory in this respect. The formal procedure employed in obtaining the above solution is justified if

$$\alpha_1,\ \alpha_2,\ \epsilon_1,\ \epsilon_2, = O(1), \qquad \sigma_1,\ \sigma_2,\ \tau_1,\ \tau_2 = o\,(u^{-3}) \text{ as } u \to \infty, \tag{106}$$

although it is obvious that we may relax these conditions considerably.

The solution simplifies considerably if we put $u^0 = v^0 = 0$, as was done by Hopkins (**4**). This is apparent in the first of the two special cases considered below, viz the solution involving specified displacements along one boundary and specified stresses along the other.

14. The infinite strip with one boundary free from displacements and the other subjected to a Gaussian distribution of normal pressure

In this case

$$u^0 = v^0 = 0, \qquad \widetilde{xy}^1 = 0, \qquad \widetilde{yy}^1 = -\frac{e^{-x^2/\delta^2}}{\delta\pi^{\frac{1}{2}}}, \tag{107}$$

where δ is a parameter. Hence

$$\alpha_1 = \alpha_2 = \epsilon_1 = \epsilon_2 = \tau_1 = \tau_2 = \sigma_2 = 0, \qquad \sigma_1 = -e^{-\delta^2u^2/4}. \tag{108}$$

From (101) $\qquad p = -\frac{\kappa\sigma_1}{\mu}, \qquad q = r = t = 0,$

and, from (102),

$$\beta_2 = 0, \qquad g\mu u\beta_1 = \kappa[2\lambda c+s(1-\kappa)]e^{-\delta^2u^2/4}, \tag{109}$$

$$\gamma_1 = 0, \qquad g\mu u\gamma_2 = \kappa[2\lambda s+c(1+\kappa)]e^{-\delta^2u^2/4}. \tag{110}$$

Thus we have $\qquad \Omega_0 = \Omega_1 = \omega_0 = \omega_1 = 0,$

and
$$\Omega(z) = \Omega_2(z)+\Omega_3(z) = \frac{2\mu i}{\pi}\int\limits_0^\infty (Ae^{izu}+Be^{-izu})\,du, \tag{111}$$

where

$$g\mu uA = [(2\lambda+1)e^{\lambda}+\kappa e^{-\lambda}]e^{-\delta^2u^2/4}, \qquad g\mu uB = [(2\lambda-1)e^{-\lambda}-\kappa e^{\lambda}]e^{-\delta^2u^2/4};$$

also

$$\omega(z) = \omega_2(z)+\omega_3(z) = -z\Omega(z)+\frac{2\mu}{\pi}\int\limits_0^\infty (Pe^{izu}+Qe^{-izu}+R)\,du, \tag{112}$$

where

$$g\mu u^2P = [(\kappa+1)(2\lambda s+c+\kappa c)+(1-\kappa)(2\lambda c+s-\kappa s)]e^{-\delta^2u^2/4},$$
$$g\mu u^2Q = [(\kappa+1)(2\lambda s+c+\kappa c)+(\kappa-1)(2\lambda c+s-\kappa s)]e^{-\delta^2u^2/4},$$
$$g\mu u^2R = -2(1+\kappa)^2.$$

15. The infinite strip with one boundary free from displacements and the other subjected to an isolated force.

Neither the solution given by Hopkins nor that described above is rigorous when $\delta = 0$, i.e. when the specified stresses have a singularity corresponding to an isolated force. A satisfactory solution in this case may be obtained in the following manner.

Potentials corresponding to an isolated force $F = X+iY$ at the point $z = ic_0$ on the boundary of elastic material are given by

$$\begin{aligned} \pi\Omega(z) &= -2F\log(z-ic_0), \\ \pi\omega(z) &= [2\bar{F}(z-ic_0)-2Fic_0]\log(z-ic_0). \end{aligned} \tag{113}$$

By the method given for the solution of problems of a half-plane (7) we find that the following potentials annul the displacements given by (113) along the boundary $y = 0$:

$$\begin{aligned}\pi\kappa\Omega(z) &= 2F\log(z+ic_0)+4ic_0\,\bar{F}/(z+ic_0),\\ \pi\kappa\omega(z) &= -\pi\kappa z\Omega(z)+[2(F-\kappa^2\bar{F})(z+ic_0)+4ic_0\,\bar{F}]\log(z+ic_0)+\\ &\qquad +2(\kappa-1)(F+\kappa\bar{F})z.\end{aligned} \tag{114}$$

The latter potentials give the following stresses along $y = c_0$:

$$\begin{aligned}-\pi\kappa\psi(x) = \pi\kappa\widetilde{yy}^1 &= \frac{Q}{2}[Xx(1-\kappa^2)+2Yc_0(\kappa^2-1)]+\\ &\quad +4c_0^2\,Q^2(Xx-2c_0\,Y)+64c_0^4\,Q^3(2c_0\,Y-Xx),\\ -\pi\kappa\chi(x) = \pi\kappa\widetilde{xy}^1 &= \frac{Q}{2}[Yx(\kappa^2-1)+2Xc_0(3+\kappa^2)]-\\ &\quad -4c_0^2\,Q^2(Yx+10c_0\,X)+64c_0^4\,Q^3(2c_0\,X+Yx),\end{aligned} \tag{115}$$

where $$Q = (x^2+4c_0^2)^{-1}.$$

Thus the sum of the above two pairs of potentials gives zero displacements along the line $y = 0$ but introduces the additional stresses given in (115) along $y = c_0$. We therefore introduce further potentials which give zero displacements along $y = 0$, have no singularities inside or on the boundary of R, and give

$$\widetilde{yy}^1 = \psi(x), \qquad \widetilde{xy}^1 = \chi(x). \tag{116}$$

Hence we require the potentials given in (94), (95) with β_1, etc., given by (102) and $\psi(x)$, $\chi(x)$ as defined in (115). In the present case

$$\begin{aligned}\alpha_1 = \alpha_2 = \epsilon_1 &= \epsilon_2 = 0,\\ -2\kappa e^{2\lambda}\sigma_1 &= Y(\kappa^2+1+4\lambda+4\lambda^2),\\ -2\kappa e^{2\lambda}\sigma_2 &= X(\kappa^2-1+4\lambda^2),\\ -2\kappa e^{2\lambda}\tau_1 &= X(\kappa^2+1-4\lambda+4\lambda^2),\\ -2\kappa e^{2\lambda}\tau_2 &= Y(1-\kappa^2+4\lambda^2).\end{aligned} \tag{117}$$

The modified potentials given in (94), (95) are required, as (104) is not satisfied; since the transforms of the boundary stress functions satisfy (89) no further modifications are required. From (101)

$$\mu p = -\kappa\sigma_1, \qquad \mu q = -\kappa\tau_1, \qquad \mu r = \kappa\sigma_2, \qquad \mu t = \kappa\tau_2, \tag{118}$$

and, from (100),

$$\frac{g\beta_1 u}{2\mu e^{2\lambda}} = Y[(\kappa^2+1+4\lambda+4\lambda^2)(2\lambda c+s-\kappa s)+(\kappa^2+4\lambda^2-1)(2\lambda s-c-\kappa c)],$$

$$\frac{g\gamma_2 u}{2\mu e^{2\lambda}} = Y[(\kappa^2+1+4\lambda+4\lambda^2)(2\lambda s+c+\kappa c)+(\kappa^2+4\lambda^2-1)(2\lambda c+\kappa s-s)],$$

$$\frac{g\beta_2 u}{2\mu e^{2\lambda}} = X[(\kappa^2+1-4\lambda+4\lambda^2)(2\lambda s-c-\kappa c)+(-\kappa^2+1-4\lambda^2)(\kappa s-s-2\lambda c)],$$

$$\frac{g\gamma_1 u}{2\mu e^{2\lambda}} = X[(\kappa^2+1-4\lambda+4\lambda^2)(s-\kappa s-2\lambda c)+(-\kappa^2+1-4\lambda^2)(c+\kappa c+2\lambda s)]. \tag{119}$$

The complete solution to the problem is given by the sum of the potentials given in (113), (114), (94), (95). Although the integrands contain more terms than do those in the potentials obtained by treating the problem as the limiting case of a Gaussian distribution, the above solution has the advantage of mathematical rigour, and, further, the integrals involved converge more rapidly. For example, the integrand in (111) is $O(e^{-\lambda})$ at $u = \infty$, uniformly in R, compared with $O(e^{-2\lambda})$ in the present case.

We now solve problems involving isolated forces in the interiors of infinite strips. Complex potentials are employed to determine the stresses and displacements given by an isolated force in the interior of an infinite plate, and the lines $y = 0, c_0$ are made either stress-free or free from displacements by the introduction of further potentials which have no singularities either inside or on the boundary of R and which annul the stresses or displacements given by the isolated force potentials along the lines $y = 0$, c_0.

Problems of Type IV

16. Formulae connected with an isolated force in the interior of an infinite plate

The potentials corresponding to an isolated force $F = X+iY$ at the point $z = b+ia$ in the interior of an infinite plate are

$$\begin{aligned}\Omega(z) &= -\nu F\log(z-b-ia),\\ \omega(z) &= \nu\kappa\bar{F}(z-b-ia)\log(z-b-ia)+\nu F(b-ia)\log(z-b-ia),\end{aligned} \tag{120}$$

where $$\nu\pi(\kappa+1) = 2.$$

From (3), (4) we find that the stresses and displacements are given by

$$\frac{4\widetilde{yy}}{\nu} = \frac{(\kappa-1)[X(x-b)-Y(y-a)]}{(x-b)^2+(y-a)^2} - \frac{4(y-a)^2[X(x-b)+Y(y-a)]}{[(x-b)^2+(y-a)^2]^2},$$

$$\frac{4\widetilde{xy}}{\nu} = \frac{(1-\kappa)Y(x-b)-(\kappa+3)X(y-a)}{(x-b)^2+(y-a)^2} - \frac{4(y-a)^2[Y(x-b)-X(y-a)]}{[(x-b)^2+(y-a)^2]^2}, \tag{121}$$

and

$$\frac{8\mu u}{\nu} = \frac{-2X(y-a)^2+2Y(x-b)(y-a)}{(x-b)^2+(y-a)^2}+(1-\kappa)X-\kappa X\log[(x-b)^2+(y-a)^2],$$

$$\frac{8\mu v}{\nu} = \frac{2Y(y-a)^2+2X(x-b)(y-a)}{(x-b)^2+(y-a)^2}-(1+\kappa)Y-\kappa Y\log[(x-b)^2+(y-a)^2]. \tag{122}$$

17. Image potentials suitable for removing the non-evanescent displacements given by the isolated force potentials

The whole of the preceding theory is dependent on the validity of the Fourier integral expressions for the boundary stress and displacement functions. This weakness, which is common to all previous methods of treatment of the infinite strip, may be removed by the addition of elementary potentials which we shall refer to as 'image potentials'. In the case of an isolated force in the interior of an infinite strip with stress-free boundaries the stresses which are to be annulled have representative functions which are expressible as Fourier integrals and the general method developed above needs no modification, but, if either boundary is to be free from displacements, we can see from (122) that the displacements to be annulled contain terms which are infinite at infinity. However, the author has shown previously (**7**) that the potentials

$$\begin{aligned}\Omega(z) &= \nu F\log(z-b+ia),\\ \omega(z) &= -z\Omega(z)+\nu(F-\kappa\bar{F})(z-b+ia)\log(z-b+ia)\end{aligned} \tag{123}$$

remove the finite and infinite terms in the displacements along $y = 0$ given by the isolated force. We see that these potentials also remove the non-evanescent displacements along $y = c_0$. From (3), (4) the stresses and displacements given by (123) are

$$\begin{aligned}\frac{4\widetilde{yy}}{\nu} &= \frac{(1-\kappa)X(x-b)+(1+\kappa)Y(y+a)-2Yy}{(x-b)^2+(y+a)^2}+\\ &\qquad +\frac{4y[Y(y+a)^2+X(x-b)(y+a)]}{[(x-b)^2+(y+a)^2]^2},\\ \frac{4\widetilde{xy}}{\nu} &= \frac{(\kappa-1)Y(x-b)+(1+\kappa)X(y+a)+2Xy}{(x-b)^2+(y+a)^2}+\\ &\qquad +\frac{4y[-X(y+a)^2+Y(x-b)(y+a)]}{[(x-b)^2+(y+a)^2]^2},\end{aligned} \tag{124}$$

and

$$\frac{8\mu u}{\nu} = \frac{2y[X(y+a)-Y(x-b)]}{(x-b)^2+(y+a)^2} - (1-\kappa)X+\kappa X\log[(x-b)^2+(y+a)^2],$$
$$\frac{8\mu v}{\nu} = \frac{-2y[X(x-b)+Y(y+a)]}{(x-b)^2+(y+a)^2} + (1+\kappa)Y+\kappa Y\log[(x-b)^2+(y+a)^2]. \tag{125}$$

18. Isolated longitudinal force in the interior of an infinite strip with the straight boundaries stress-free

We solve the problem of an isolated force $F = X+0i$ at the point $z = b+ia$ in the interior of the strip $0 \leqslant y \leqslant c_0$ with the boundaries $y = 0$, c_0 stress-free. The solution to a similar problem was first given by Howland (3). From (121) we see that the potentials additional to those given in (120) must satisfy the boundary conditions

$$\begin{aligned}
\frac{4\widetilde{yy}^0}{\nu X} &= \frac{(1-\kappa)(x-b)}{(x-b)^2+a^2} + \frac{4a^2(x-b)}{[(x-b)^2+a^2]^2},\\
\frac{4\widetilde{xy}^0}{\nu X} &= \frac{-(\kappa+3)a}{(x-b)^2+a^2} + \frac{4a^3}{[(x-b)^2+a^2]^2},\\
\frac{4\widetilde{yy}^1}{\nu X} &= \frac{(1-\kappa)(x-b)}{(x-b)^2+(c_0-a)^2} + \frac{4(c_0-a)^2(x-b)}{[(x-b)^2+(c_0-a)^2]^2},\\
\frac{4\widetilde{xy}^1}{\nu X} &= \frac{(\kappa+3)(c_0-a)}{(x-b)^2+(c_0-a)^2} - \frac{4(c_0-a)^3}{[(x-b)^2+(c_0-a)^2]^2}.
\end{aligned} \tag{126}$$

Thus we require the solution given in section 2 of this paper. With the notation given in (5), (7), (9), (10) we have

$$\begin{aligned}
4\alpha_1 &= \pi\nu X e^{-au}(\kappa-1-2au)\sin bu,\\
4\alpha_2 &= \pi\nu X e^{-au}(\kappa-1-2au)\cos bu,\\
4\epsilon_1 &= -\pi\nu X e^{-au}(\kappa+1-2au)\cos bu,\\
4\epsilon_2 &= \pi\nu X e^{-au}(\kappa+1-2au)\sin bu,\\
4\sigma_1 &= \pi\nu X e^{-u(c_0-a)}[\kappa-1-2u(c_0-a)]\sin bu,\\
4\sigma_2 &= \pi\nu X e^{-u(c_0-a)}[\kappa-1-2u(c_0-a)]\cos bu,\\
4\tau_1 &= \pi\nu X e^{-u(c_0-a)}[\kappa+1-2u(c_0-a)]\cos bu,\\
4\tau_2 &= -\pi\nu X e^{-u(c_0-a)}[\kappa+1-2u(c_0-a)]\sin bu.
\end{aligned} \tag{127}$$

Now

$$\alpha_1,\, \alpha_2,\, \epsilon_1,\, \epsilon_2 = O(ue^{-au}), \qquad \sigma_1,\, \sigma_2,\, \tau_1,\, \tau_2 = O(ue^{-u(c_0-a)}) \text{ as } u \to \infty, \tag{128}$$

and, since $0 < a < c_0$, (29) is satisfied and all integrals are uniformly convergent with respect to z in the region R. Using the notation of (54)–(57) we can see from (25) that in this case

$$\begin{gathered}
A_1 = 3X(c_0-2a)/c_0^3, \qquad N_1 = X(2c_0-3a)/c_0^2,\\
B_1 = B_2 = M_1 = M_2 = 0.
\end{gathered} \tag{129}$$

The stresses at $x = \pm\infty$ are respectively

$$\widetilde{yy} = \widetilde{xy} = 0, \qquad \widetilde{xx} = \pm X[3y(c_0-2a)-c_0(2c_0-3a)]/c_0^3. \tag{130}$$

each of which is statically equivalent to the force wrench

$$\{-X/2,\ 0,\ Xa/2\}. \tag{131}$$

Fig. 2 shows the external forces on the strip in the case $3a/2 < c_0 < 2a$.

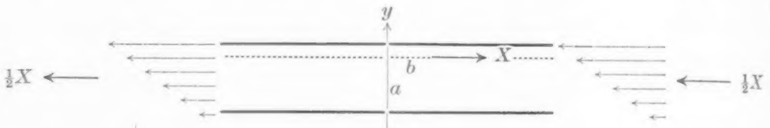

FIG. 2.

The force resultants at $x = \pm\infty$ may be modified by the superposition of simple tensions parallel to the x-axis.

19. Isolated transverse force in the interior of an infinite strip with the straight boundaries stress-free

If an isolated force $F = 0+iY$ is applied at the point $z = b+ia$ in the interior of an infinite strip $0 \leqslant y \leqslant c_0$, then we can see from (121) that the additional potentials required to make the straight boundaries stress-free must satisfy the boundary conditions

$$\begin{aligned}
\frac{4\widetilde{yy}^0}{\nu Y} &= \frac{(1-\kappa)a}{(x-b)^2+a^2}-\frac{4a^3}{[(x-b)^2+a^2]^2},\\
\frac{4\widetilde{xy}^0}{\nu Y} &= \frac{(\kappa-1)(x-b)}{(x-b)^2+a^2}+\frac{4a^2(x-b)}{[(x-b)^2+a^2]^2},\\
\frac{4\widetilde{yy}^1}{\nu Y} &= \frac{(\kappa-1)(c_0-a)}{(x-b)^2+(c_0-a)^2}+\frac{4(c_0-a)^3}{[(x-b)^2+(c_0-a)^2]^2},\\
\frac{4\widetilde{xy}^1}{\nu Y} &= \frac{(\kappa-1)(x-b)}{(x-b)^2+(c_0-a)^2}+\frac{4(x-b)(c_0-a)^2}{[(x-b)^2+(c_0-a)^2]^2}.
\end{aligned} \tag{132}$$

We again use the solution given in section 2. With the notation of (5), (7), (9), (10) we have

$$\begin{aligned}
4\alpha_1 &= -\pi\nu Ye^{-au}(\kappa+1+2au)\cos bu,\\
4\alpha_2 &= \pi\nu Ye^{-au}(\kappa+1+2au)\sin bu,\\
4\epsilon_1 &= -\pi\nu Ye^{-au}(\kappa-1+2au)\sin bu,\\
4\epsilon_2 &= -\pi\nu Ye^{-au}(\kappa-1+2au)\cos bu,\\
4\sigma_1 &= \pi\nu Ye^{-u(c_0-a)}[\kappa+1+2u(c_0-a)]\cos bu,\\
4\sigma_2 &= -\pi\nu Ye^{-u(c_0-a)}[\kappa+1+2u(c_0-a)]\sin bu,\\
4\tau_1 &= -\pi\nu Ye^{-u(c_0-a)}[\kappa-1+2u(c_0-a)]\sin bu,\\
4\tau_2 &= -\pi\nu Ye^{-u(c_0-a)}[\kappa-1+2u(c_0-a)]\cos bu.
\end{aligned} \tag{133}$$

The orders of magnitude of these functions at $u = \infty$ are the same as in the previous case and are, therefore, satisfactory. With the notation of (25), (54)–(57) we have in the present case

$$\begin{aligned} &A_1 = 6bY/c_0^3, \qquad B_1 = 6Y/c_0^3, \qquad M_1 = -3Y/c_0^2, \\ &N_1 = 3bY/c_0^2, \qquad B_2 = M_2 = 0. \end{aligned} \tag{134}$$

The stresses at $x = \pm\infty$ are respectively

$$\widetilde{yy} = 0, \qquad \widetilde{xy} = \pm 3Yy(y-c_0)/c_0^3, \qquad \widetilde{xx} = \lim_{x\to\pm\infty} [3Y(c_0-2y)(|x| \mp b)/c_0^3], \tag{135}$$

which are statically equivalent to the wrenches

$$\lim_{x\to\pm\infty} \{0,\ -Y/2,\ Y(x-b)/2\} \tag{136}$$

which are shown in Fig. 3.

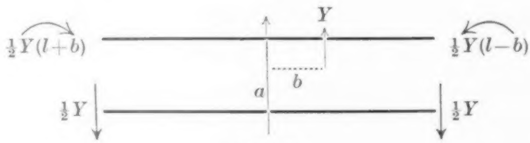

FIG. 3.

The above solution reduces to that of Howland (**3**) when the constant b is equated to zero.

20. Isolated force in the interior of an infinite strip with the straight boundaries free from displacements

In this case we begin with the isolated force potentials given in (120) and the image potentials given in (123). From (122), (125) we see that, in order to free the straight boundaries from displacements, we require potentials which satisfy the following boundary conditions:

$$4\mu u^0/\nu = \frac{aY(x-b)+a^2xX}{(x-b)^2+a^2}, \qquad 4\mu v^0/\nu = \frac{aX(x-b)-a^2Y}{(x-b)^2+a^2}, \tag{137}$$

$$\begin{aligned} 4\mu u^1/\nu = {} & \frac{X(c_0-a)^2-Y(c_0-a)(x-b)}{(x-b)^2+(c_0-a)^2} + \\ & + \frac{-c_0X(c_0+a)+c_0Y(x-b)}{(x-b)^2+(c_0+a)^2} - \frac{\kappa X}{2}\log\frac{(x-b)^2+(c_0+a)^2}{(x-b)^2+(c_0-a)^2}, \end{aligned} \tag{138}$$

$$\begin{aligned} 4\mu v^1/\nu = {} & \frac{-X(c_0-a)(x-b)-Y(c_0-a)^2}{(x-b)^2+(c_0-a)^2} + \\ & + \frac{c_0X(x-b)+c_0Y(c_0+a)}{(x-b)^2+(c_0+a)^2} - \frac{\kappa Y}{2}\log\frac{(x-b)^2+(c_0+a)^2}{(x-b)^2+(c_0-a)^2}. \end{aligned} \tag{139}$$

We require the solution given in section 12 and, with the notation of (5), (7), (78), (84), we have

$$\begin{aligned} 4\mu e^{au}\alpha_1 &= \nu a\pi(X\cos bu - Y\sin bu), \\ 4\mu e^{au}\alpha_2 &= \nu a\pi(-X\sin bu - Y\cos bu), \\ 4\mu e^{au}\epsilon_1 &= \nu a\pi(-X\sin bu - Y\cos bu), \\ 4\mu e^{au}\epsilon_2 &= \nu a\pi(-X\cos bu + Y\sin bu), \end{aligned} \tag{140}$$

$$\begin{aligned} \frac{4\mu\sigma_1 e^{\lambda}}{\nu\pi} &= \left[2\left(c_0 - \frac{\kappa}{u}\right)\sinh au - ae^{au}\right]X\cos bu + [2c_0\sinh au - ae^{au}]Y\sin bu, \\ \frac{4\mu\sigma_2 e^{\lambda}}{\nu\pi} &= \left[ae^{au} - 2\left(c_0 - \frac{\kappa}{u}\right)\sinh au\right]X\sin bu + [2c_0\sinh au - ae^{au}]Y\cos bu, \\ \frac{4\mu\tau_1 e^{\lambda}}{\nu\pi} &= [2c_0\sinh au - ae^{au}]X\sin bu + \left[ae^{au} - 2\left(c_0 + \frac{\kappa}{u}\right)\sinh au\right]Y\cos bu, \\ \frac{4\mu\tau_2 e^{\lambda}}{\nu\pi} &= [2c_0\sinh au - ae^{au}]X\cos bu + \left[2\left(c_0 + \frac{\kappa}{u}\right)\sinh au - ae^{au}\right]Y\sin bu. \end{aligned} \tag{141}$$

From (140), (141) we can see that (89) is satisfied at $u = 0$, but that (90) is not. However,

$$\begin{aligned} \alpha_2 - \sigma_2 &= aY(e^{au} - 1)\cos bu + O(u) = O(u), \\ \epsilon_2 - \tau_2 &= aX(e^{au} - 1)\cos bu + O(u) = O(u) \quad \text{as } u \to 0. \end{aligned}$$

Thus (91) is satisfied and the modifications developed above are not necessary in this case. Also, since (128) is satisfied, all integrals are uniformly convergent in the region R. Satisfactory additional potentials are therefore given by (79)–(82).

21. Isolated force in the interior of an infinite strip with one boundary free from displacements and the other stress-free

We again begin with the isolated force potentials and the image potentials. In order to free the boundary $y = 0$ from displacements and the boundary $y = c_0$ from stresses we require potentials which satisfy the conditions given in (137) along $y = 0$ and, from (121), (124), give along $y = c_0$

$$\begin{aligned} 4\widetilde{yy}^1/\nu &= \frac{(1-\kappa)[X(x-b) - Y(c_0-a)]}{(x-b)^2 + (c_0-a)^2} + \frac{4(c_0-a)^2[X(x-b) + Y(c_0-a)]}{[(x-b)^2 + (c_0-a)^2]^2} + \\ &\quad + \frac{(\kappa-1)X(x-b) - (\kappa+1)Y(c_0+a) + 2c_0 Y}{(x-b)^2 + (c_0+a)^2} - \frac{4c_0(c_0+a)[X(x-b) + Y(c_0+a)]}{[(x-b)^2 + (c_0+a)^2]^2}, \\ 4\widetilde{xy}^1/\nu &= \frac{(\kappa-1)Y(x-b) + (\kappa+3)X(c_0-a)}{(x-b)^2 + (c_0-a)^2} + \frac{4(c_0-a)^2[Y(x-b) - X(c_0-a)]}{[(x-b)^2 + (c_0+a)^2]^2} - \\ &\quad - \frac{(\kappa-1)Y(x-b) - (\kappa+1)X(c_0+a) - 2c_0 X}{(x-b)^2 + (c_0+a)^2} - \frac{4c_0(c_0+a)[Y(x-b) - X(c_0+a)]}{[(x-b)^2 + (c_0+a)^2]^2}. \end{aligned} \tag{142}$$

We require the solution given in section 13 and, with the notation of (5), (7), (10), (78), we see that α_1, α_2, ϵ_1 ϵ_2 are given by (140), whilst σ_1, σ_2, τ_1, τ_2 are given by

$$\begin{aligned}
\frac{2\sigma_1 e^{\lambda}}{\nu\pi} &= [aue^{au}+(\kappa-1-2\lambda)\sinh au]X\sin bu+\\
&\qquad +[-aue^{au}+(1+\kappa+2\lambda)\sinh au]Y\cos bu,\\
\frac{2\sigma_2 e^{\lambda}}{\nu\pi} &= [aue^{au}+(\kappa-1-2\lambda)\sinh au]X\cos bu+\\
&\qquad +[aue^{au}-(1+\kappa+2\lambda)\sinh au]Y\sin bu,\\
\frac{2\tau_1 e^{\lambda}}{\nu\pi} &= [aue^{au}+(1+\kappa-2\lambda)\sinh au]X\cos bu+\\
&\qquad +[aue^{au}+(1-\kappa-2\lambda)\sinh au]Y\sin bu,\\
\frac{2\tau_2 e^{\lambda}}{\nu\pi} &= [-aue^{au}-(1+\kappa-2\lambda)\sinh au]X\sin bu+\\
&\qquad +[aue^{au}-(\kappa-1+2\lambda)\sinh au]Y\cos bu.
\end{aligned} \tag{143}$$

In this case (104) is not satisfied but, since (89) is satisfied, the modifications obtained above are necessary and sufficient to ensure that the integrals are satisfactory at $u = 0$, whilst conditions at $u = \infty$ are the same as in the previous case. The solution to the problem is given by the sum of the isolated force potentials, the image potentials, and the potentials given in (79), (80), (94), (95).

The author wishes to take this opportunity of expressing his gratitude to A. C. Stevenson for the advice which he has freely given.

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NEUTRAL HOLES IN PLANE SHEET—REINFORCED HOLES WHICH ARE ELASTICALLY EQUIVALENT TO THE UNCUT SHEET

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SUMMARY

It is shown that, in a plane sheet under any particular loading system, acting in the plane of the sheet, certain reinforced holes may be made which do not alter the stress distribution in the main body of the sheet. These reinforced holes (hereafter called neutral holes) necessarily have exactly the same stiffness and at least the same strength as the portion of the sheet that has been cut out. The weight of the reinforcement is usually greater than the weight of the sheet that has been cut out, though there are cases where it is less.

Mathematical formulae are developed to determine both the shape of a neutral hole and the variation along the hole boundary of the cross-sectional area of the reinforcement.

List of symbols

Ox, Oy	Cartesian coordinate axes.
t	thickness of sheet.
σ_x, σ_y, τ_{xy}	stresses in the sheet.
ϕ	Airy stress function (see equation (1)).
ψ	angle made by the tangent to boundary of hole with Ox.
P	load in reinforcing member bounding hole.
A_m	section area of reinforcing member bounding hole.
A_j	section area of straight reinforcing member.
ϵ	strain in reinforcing member bounding hole.
E	Young's modulus.
ν	Poisson's ratio.

1. General properties of a neutral hole

CONSIDER first the sheet in the uncut state. The stresses are such that all elements of the sheet are in equilibrium which, for a typical element of the sheet, requires that

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y} = 0$$

and

$$\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{xy}}{\partial x} = 0.$$

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Both these relations are automatically satisfied by introducing a stress function ϕ such that the stresses are to be derived from it by the equation

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2\phi}{\partial y^2} \\ \sigma_y &= \frac{\partial^2\phi}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2\phi}{\partial x \partial y} \end{aligned}\right\}. \tag{1}$$

The complete state of stress in a sheet, therefore, can be described by the stress-function ϕ alone. This stress function is usually introduced as an aid to the determination of the stresses in a plate subjected to given boundary conditions. For example, for a plain sheet it can be shown that ϕ satisfies a particular equation ($\nabla^4\phi = 0$) which, together with the boundary conditions, is sufficient to determine ϕ, and hence the stresses. Here, however, it is assumed that the complete stress distribution is already known, but ϕ will still be used because it admits of great generality and because the properties of a neutral hole can be expressed simply in terms of ϕ and its derivatives. The function ϕ itself assumes a new and special significance in determining the shape of a neutral hole.

1.1. The shape of a neutral hole

To fix ideas whilst considering the equilibrium of the reinforcing member and the adjacent sheet Figs. 1 and 2 are given below.

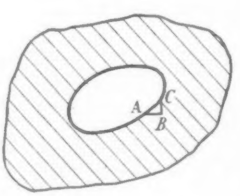


FIG. 1. Sheet with neutral hole.

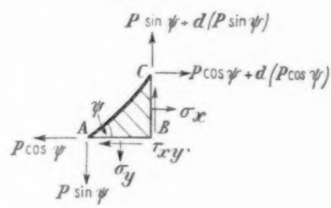


FIG. 2. Forces acting on the element ABC.

It can be shown that, in a practical construction, the bending stiffness of the reinforcing member is negligible compared with the tensile stiffness so that the reinforcing member has the properties of a chain in that the line of action of the load P in the member is directed along the length of the member. For the element shown in Fig. 2 the conditions of equilibrium are therefore

$$\left.\begin{aligned} d(P\sin\psi) &= t(\sigma_y\,dx - \tau_{xy}\,dy) \\ \text{and} \quad d(P\cos\psi) &= t(\tau_{xy}\,dx - \sigma_x\,dy) \end{aligned}\right\}. \tag{2}$$

Equations (1) and (2) can be combined to give

$$\left.\begin{aligned}(1/t)d(P\sin\psi) &= \frac{\partial^2\phi}{\partial x^2}dx+\frac{\partial^2\phi}{\partial x\partial y}dy\\ \text{and}\qquad (1/t)d(P\cos\psi) &= \frac{-\partial^2\phi}{\partial x\partial y}dx-\frac{\partial^2\phi}{\partial y^2}dy\end{aligned}\right\}. \tag{3}$$

These equations are in the form of total differentials and may therefore be integrated to give

$$\left.\begin{aligned}\frac{P\sin\psi}{t} &= \frac{\partial\phi}{\partial x}+a\\ \frac{P\cos\psi}{t} &= -\frac{\partial\phi}{\partial y}-b\end{aligned}\right\}, \tag{4}$$

where a and b are arbitrary constants.

The load P may be eliminated from equation (4) to give

$$\tan\psi = -\left[\frac{\partial\phi/\partial x+a}{\partial\phi/\partial y+b}\right]. \tag{5}$$

Substituting dy/dx for $\tan\psi$, equation (5) may be integrated to give

$$\phi+ax+by+c = 0 \tag{6}$$

as the equation for determining the shape of a neutral hole.

Terms of the type $(ax+by+c)$ can be added to ϕ without altering the stresses and so there will be no loss of generality by writing equation (6) as

$$\phi = 0. \tag{7}$$

Hereafter equation (7) will be used instead of equation (6) as this appreciably simplifies the presentation of results. It is worth noting that equation (7) is derived purely from considerations of statics and is independent of the elastic properties of the sheet.

1.11. Hole bounded by arcs

Since there are no restrictions on the constants a, b, c that may be included in ϕ it will be seen that there is a large variety of curves from which the hole shape may be chosen. Furthermore, the hole shape may be bounded by arcs of curves, each of which is determined by a different set of values of a, b, c; in this case it will be necessary to apply balancing loads at the junction points of adjacent arcs to ensure equilibrium of the loads in the reinforcing members. It can be shown that such balancing loads can normally be produced by inserting a simple tension or compression member from one junction point to another.

The calculation of such balancing loads is straightforward. Suppose arc 1 is given by

$$\phi_1 = \phi'+a_1x+b_1y+c = 0$$

and arc 2 by

$$\phi_2 = \phi'+a_2x+b_2y+c_2 = 0;$$

then from equation (4) the vertical (or y) component of the load in arcs 1 and 2 is $t(\partial\phi'/\partial x+a_1)$ and $t(\partial\phi'/\partial x+a_2)$, respectively, and there are similar expressions for the x-components. It follows that the vertical and horizontal components of the balancing load are respectively

$$\left.\begin{array}{ll} & t(a_1-a_2) \\ \text{and} & t(b_2-b_1) \end{array}\right\}. \tag{8}$$

1.2. Section area of the reinforcing member

The section area of the reinforcing member can be determined from a knowledge of the load P and the strain ϵ in the reinforcing member by the relation

$$A_m = P/E\epsilon;$$

P is determined from equation (4) by eliminating ψ, giving

$$P = t\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]^{\frac{1}{2}}. \tag{9}$$

From elementary theory the strain ϵ is related to the sheet stresses as follows:

$$E\epsilon = \cos^2\psi(\sigma_x-\nu\sigma_y)+\sin^2\psi(\sigma_y-\nu\sigma_x)+2\sin\psi\cos\psi(1+\nu)\tau_{xy}. \tag{10}$$

Equations (9) and (10) can be combined with (1) and (7) to give

$$A_m = t\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]^{\frac{3}{2}}\left[\frac{\partial^2\phi}{\partial x^2}\left(\frac{\partial\phi}{\partial x}\right)^2+\frac{\partial^2\phi}{\partial y^2}\left(\frac{\partial\phi}{\partial y}\right)^2+\frac{2\partial^2\phi}{\partial x\partial y}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}- \right.$$

$$\left.-\nu\left\{\frac{\partial^2\phi}{\partial x^2}\left(\frac{\partial\phi}{\partial y}\right)^2+\frac{\partial^2\phi}{\partial y^2}\left(\frac{\partial\phi}{\partial x}\right)^2-\frac{2\partial^2\phi}{\partial x\partial y}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\right\}\right]^{-1}. \tag{11}$$

2. Examples

2.1. General case of uniform stress distribution

If the axes Ox, Oy are chosen parallel to the directions of the principal stresses f_1, f_2, the stress function is given by

$$\phi = \tfrac{1}{2}(f_1 y^2+f_2 x^2)+ax+by+c$$

and so the neutral hole is in the form of a conic. Furthermore, if f_1 and f_2 are of the same sign the conic is an ellipse with lengths of axes in the ratio $\sqrt{(f_1/f_2)}$. The constants a, b, c merely determine the position and size of the hole.

2.11. Equal stress in x- and y-directions

Such a stress distribution occurs, for example, in a thin-walled sphere subjected to hydrostatic pressure. The most general form for ϕ is

$$\phi = \tfrac{1}{2}f(x^2+y^2)+ax+by+c,$$

so that the shape of the neutral hole ($\phi = 0$) is circular. The fact that the constants a, b, c are at present arbitrary means that the circle can be chosen

to have any radius and to be situated anywhere. If we take the centre of the circle at the origin and let it have a radius r we can substitute

$$\phi = \tfrac{1}{2}f(x^2+y^2-r^2) = 0$$

in equations (11) and (9) to find

$$\left.\begin{aligned} A_m &= \frac{rt}{1-\nu} \\ \text{and} \qquad P &= frt \end{aligned}\right\}. \tag{12}$$

These properties of a neutral circular hole could, of course, have been deduced simply from first principles, though it is interesting to note that the circular hole is the only possible hole; furthermore the circular hole, with this value for A_m, will not be neutral for any other stress distribution.

There is no need to limit the number of such holes to one, in fact there can be two or more overlapping, as mentioned in section 1.11, provided there is inserted a tension or compression member to ensure continuity of load P and compatibility of displacements. Two examples will demonstrate this.

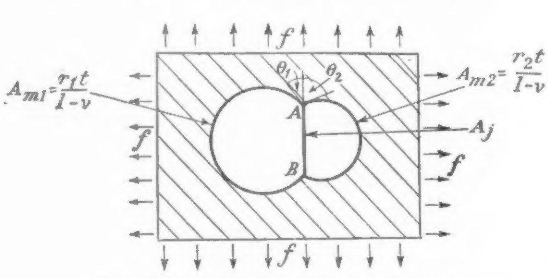


FIG. 3. Two circular holes overlapping.

In the first example, shown in Fig. 3, two unequal circular holes of radii r_1 and r_2, whose centres lie on the x-axis, overlap, meeting at an angle $(\theta_1+\theta_2)$. It follows from equation (8) that the horizontal components of P_1 and P_2 at the junction points A and B will cancel out, but the resultant vertical component $(P_1 \cos\theta_1 + P_2 \cos\theta_2)$ will necessitate a tension member between AB with a section area determined by the relation

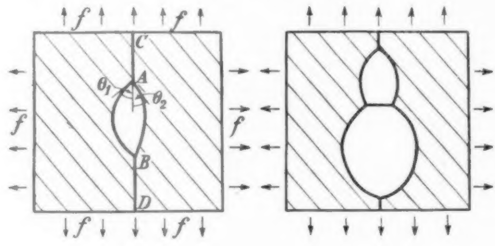
$$A_j = \frac{t}{1-\nu}(r_1 \cos\theta_1 + r_2 \cos\theta_2).$$

A different arrangement is shown in Fig. 4 (a); in this case the sign of the balancing loads at A, B is reversed and so (because members with negative section areas are not practicable) the members AC and BD must be present. These will again have a section area of

$$\frac{t}{1-\nu}(r_1 \cos\theta_1 + r_2 \cos\theta_2)$$

and loads of magnitude $ft(r_1 \cos\theta_1 + r_2 \cos\theta_2)$ must now be applied externally to them.

Fig. 4 (b) is a combination of the types represented in Figs. 3 and 4 (a).



(a) (b)

FIG. 4. Holes bounded by circular arcs.

2.12. Stress in y-direction double that in x-direction

Such a stress distribution occurs in a thin-walled cylinder subjected to hydrostatic pressure. Suppose

$$\sigma_y = 2\sigma_x = f,$$

then ϕ is given by

$$\phi = \tfrac{1}{2}f(x^2+\tfrac{1}{2}y^2)+ax+by+c,$$

so that the shape for the neutral hole is an ellipse with major and minor axes in the ratio $\sqrt{2}:1$. If a, b, c are chosen so that the centre of the ellipse is at the origin and the minor axis is $2r$, then the equation for the neutral hole will be

$$\phi \propto x^2+\tfrac{1}{2}y^2-r^2 = 0. \tag{13}$$

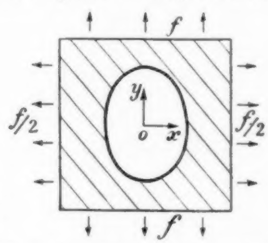


FIG. 5. Elliptical hole with axes $\sqrt{2}:1$.

Substituting equation (13) in (11) gives

$$\frac{A_m}{rt} = \frac{\sqrt{2}(1+x^2/r^2)^{\frac{3}{2}}}{1-2\nu+3x^2/r^2}. \tag{14}$$

2.13. Uniform stress in y-direction alone

If $\sigma_y = f$, the most general form for ϕ is

$$\phi = \tfrac{1}{2}fx^2+ax+by+c,$$

and so the neutral hole (or cut-out, since the hole is not closed) is any parabola of the type

$$\phi \propto x^2-ry = 0, \tag{15}$$

where r is arbitrary and determines the size.

This parabolic shape is well known in connexion with the design of

certain suspension bridges in which the weight per unit length of span, i.e. $t\sigma_y$, is constant.

The section area A_m is obtained by substituting equation (15) in equation (11); we find

$$\frac{A_m}{rt} = \frac{(1+4y/r)^{\frac{3}{2}}}{2(4y/r-\nu)}. \tag{16}$$

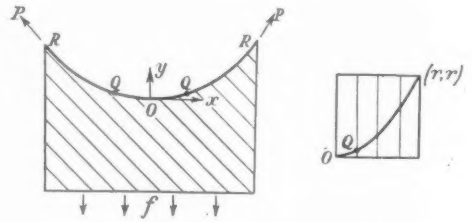


FIG. 6. Ideal parabolic cut-out.

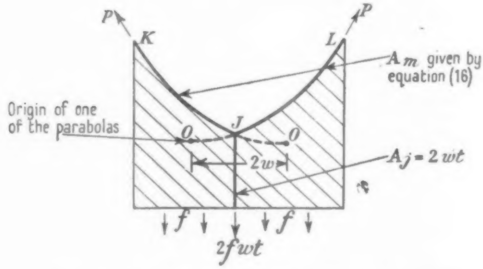


FIG. 7. Cut-out bounded by parabolic arcs.

This expression becomes negative over the range $0 < y < \nu r/4$, from Q to Q (say) in Fig. 6, and so any practical design must utilize those parts of the parabola from Q to R. In fact, as A_m is very large in the immediate vicinity of Q it will also be inadvisable, from a weight point of view, to use those parts of the parabola near Q. The inset in Fig. 6 shows the position of Q to scale taking $\nu = \frac{1}{4}$. It will be seen that the 'useless' region is comparatively small; it is due entirely to the Poisson contraction.

The simplest symmetrical arrangement with positive A_m everywhere would be that shown in Fig. 7.

At the junction point J the horizontal components of the P's will cancel out, but to offset the resultant vertical component $2fwt$, given by equation (8), it will be necessary to introduce the tension member as shown. If only vertical external forces can be applied at K, L the necessary horizontal components could be produced by inserting a compression member of section area $rt/2\nu$ between K and L.

2.2. Equal bending stresses in two directions

The most general form for ϕ is now

$$\phi \propto x^3+y^3+ax+by+c,$$

so that the shape of a neutral hole is, in general, a cubic.

Owing to the more complex form for ϕ the coefficients a, b, c no longer refer directly to the position and size of the hole. Here, however, the coefficients will be chosen specially so that the stress function may be factorized. The advantage of this is that it will be possible to find a 'closed form' for the shape of the neutral hole. We take

$$\phi \propto x^3+y^3-r^2(x+y) \equiv (x+y)(x^2-xy+y^2-r^2), \tag{17}$$

and choose the shape of the neutral hole from the second factor only.

This factor represents an ellipse with major and minor axes in the ratio $\sqrt{3}:1$ and inclined at 45° to the x-axis.

The cross-section area of the reinforcement, obtained from equations (17) and (11), is given by

$$A_m = \frac{t(5r^2-3xy)^{\frac{3}{2}}}{6\{(4-\nu)r^2-3xy\}}, \tag{18}$$

which remains practically constant at its mean value of $0{\cdot}50rt$.

2.3. Stress distributions with circular neutral holes and constant A_m

It was pointed out in section 2.11 that the circular hole with the value for A_m given by equation (12) would be neutral for one particular stress distribution and none other. But the circular shape itself is not confined to that one particular stress distribution. For example, confining attention to plain sheet, any function which satisfies

$$\nabla^4\phi = 0, \tag{19}$$

gives rise to a possible stress distribution. If, therefore, we take

$$\phi = (x^2+y^2-r^2)F(x,y) \tag{20}$$

and choose $F(x,y)$ so that equation (19) is satisfied, then a stress system will be formed for which the circle is a possible shape for a neutral hole. The section area of the reinforcing member will usually not be constant, but there is a set of possible functions of $F(x,y)$ in which A_m is constant. This set is characterized by $F(x,y)$ being homogeneous in x and y. It is more convenient to use polar coordinates in discussing this set. Thus, if c is the radius of the circle and n is the order of homogeneity of $F(x,y)$, equation (20) becomes

$$\phi = (r^2-c^2)r^nf(\theta), \tag{21}$$

and it is known that to satisfy equation (19)

$$f(\theta) = a\sin n\theta + b\cos n\theta. \tag{22}$$

Substituting in equation (11) it will be found that

$$A_m = \frac{ct}{(2n+1-\nu)}, \tag{23}$$

which is a constant; so that, except for the special case with $n = 0$ considered in section 2.11 the weight of reinforcement inserted is always appreciably less than the weight of material removed. The reason for this lies in the fact that the sheet which is to be removed is comparatively lightly stressed.

MEAN DISPLACEMENTS ON THE BOUNDARY OF AN ELASTIC SOLID

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SUMMARY

This paper is an extension of the method of mean strains introduced by Betti (**1**), many examples of which are given by Chree (**2**). The fundamental theorem of the method is proved more directly. It is used to determine the total end force on a cylindrical elastic body when the displacements on the plane ends are known and to solve the analogous problem in plane strain.

1. Theorem

In any elastic body of volume V bounded by a surface S, throughout which the components of strain are continuous,

$$\frac{1}{2}\int\limits_S (u_j n_i + u_i n_j)\, dS = \frac{1+\nu}{E}\int\limits_S x_j \overset{(n)}{T_i}\, dS - \frac{\nu}{E}\delta_{ij}\int\limits_S x_k \overset{(n)}{T_k}\, dS -$$

$$-\frac{1+\nu}{E}\int\limits_V \rho x_j(f_i - X_i)\, d\tau + \frac{\nu}{E}\delta_{ij}\int\limits_V \rho x_k(f_k - X_k)\, d\tau. \quad (1)$$

The quantities with suffixes are Cartesian tensors, the suffixes taking the values 1, 2, and 3. The notation is:

u_i = displacement,
n_i = direction cosines of outward normal to element of surface dS,
x_i = coordinate,
p_{ij} = stress tensor,
e_{ij} = strain tensor,
$\overset{(n)}{T_i}$ = stress; force per unit area acting on a surface with normal n_i,
f_i = acceleration,
X_i = body force per unit mass,
δ_{ij} = Kronecker delta = 1 if $i = j$, = 0 if $i \neq j$,
ν = Poisson's ratio,
E = Young's modulus,
ρ = density.

Proof. By Green's lemma

$$\int\limits_S u_j n_i\, dS = \int\limits_V \frac{\partial}{\partial x_i} u_j\, d\tau.$$

Hence

$$\frac{1}{2}\int\limits_S (u_j n_i + u_i n_j)\, dS = \int\limits_V \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) d\tau$$

$$= \int\limits_V e_{ij}\, d\tau, \quad \text{by definition of strain,} \tag{2}$$

$$= \int\limits_V \left(\frac{1+\nu}{E} p_{ij} - \frac{\nu}{E} p_{kk}\delta_{ij}\right) d\tau, \quad \text{by the generalized Hooke's law,}$$

$$= \frac{1+\nu}{E}\int\limits_V p_{ij}\, d\tau - \frac{\nu}{E}\delta_{ij}\int\limits_V p_{kk}\, d\tau. \tag{3}$$

Again by Green's lemma and using $\overset{(n)}{T}_i = p_{ij} n_j$,

$$\int\limits_S x_j \overset{(n)}{T}_i\, dS = \int\limits_S x_j p_{ik} n_k\, dS = \int\limits_V \frac{\partial}{\partial x_k}(x_j p_{ik})\, d\tau$$

$$= \int\limits_V \delta_{jk} p_{ik}\, d\tau + \int\limits_V x_j \frac{\partial}{\partial x_k} p_{ik}\, d\tau$$

$$= \int\limits_V p_{ij}\, d\tau + \int\limits_V \rho x_j (f_i - X_i)\, d\tau \tag{4}$$

since, by the equations of motion,

$$\frac{\partial}{\partial x_k} p_{ik} + \rho X_i = \rho f_i.$$

Substituting for $\int\limits_V p_{ij}\, d\tau$ from (4) in (3) proves the theorem. Equating the right-hand sides of (1) and (2) gives the mean strain theorem.

This theorem cannot be directly applied to general curvilinear orthogonal coordinates, since the components of the transformation matrix from the cartesian to the general tensor will in general depend upon the position of the element dS. However, if summation takes place over the indices i and j, the integrand of each integral is an invariant and the equation will be true for any system of curvilinear orthogonal coordinates.†

† Chree proves the mean strain theorem from the reciprocal theorem

$$\int\limits_V e_{ij} p'_{ij}\, d\tau = \int\limits_V e'_{ij} p_{ij}\, d\tau,$$

where the accented and unaccented symbols refer to two different stress systems for a particular elastic body. This reciprocal theorem is true for any curvilinear orthogonal coordinate system since the integrands are invariants. To prove the mean strain theorem a particular p'_{ij} is chosen equal to 1 and the other p'_{ij} to 0. Such is a possible stress system for cartesian coordinates but it is not so in general. For example, in both polar and cylindrical coordinates the stress system defined by $\widehat{r\theta} = 1$, other stress components zero, violates the equation of equilibrium in the θ-direction when neither body forces nor accelerations are present.

Equation (1) consists in fact of six equations, obtained by giving all possible values to i and j. In the examples to follow, body forces and accelerations are absent. Equation (1) then becomes

$$\int\limits_S u_1 n_1\, dS = \frac{1+\nu}{E}\int\limits_S x_1 \overset{(n)}{T}_1\, dS - \frac{\nu}{E}\int\limits_S x_k \overset{(n)}{T}_k\, dS, \tag{5}$$

$$\frac{1}{2}\int\limits_S (u_2 n_3 + u_3 n_2)\, dS = \frac{1+\nu}{E}\int\limits_S x_2 \overset{(n)}{T}_3\, dS = \frac{1+\nu}{E}\int\limits_S x_3 \overset{(n)}{T}_2\, dS, \tag{6}$$

and four more equations obtained by permutation of the indices 1, 2, and 3 in equations (5) and (6).

The total force $\bar{Q}_i$ over a part $\bar{S}$ of the boundary surface S is given by

$$\bar{Q}_i = \int\limits_{\bar{S}} \overset{(n)}{T}_i\, dS. \tag{7}$$

2. Compression of a cylinder

Consider a cylinder of any cross-section whose plane ends are normal to the generators. It is compressed between two smooth rigid surfaces, which are mirror images of one another in the central cross-section of the cylinder, but otherwise are only limited in shape by having continuously turning tangent planes at all points and such that all normals to the surfaces make small angles with the generators.

Let the central cross-section be taken as the plane $x_1 = 0$. If the height of the cylinder is $2a$ the end planes are $x_1 = \pm a$. When the ends are just in contact with the rigid surfaces all over the ends, the displacement u_1 at each point on the end is known from the geometry of the rigid surface.† On the end $x_1 = a$, $n_1 = +1$; on the end $x_1 = -a$, $n_1 = -1$; and on the curved surface of the cylinder, $n_1 = 0$.

Since the displacement u_1 is odd in x_1 the value of the left-hand side of (5) is known and is equal to $2\int\limits_{x_1=a} u_1\, dS$. If no forces are applied to the sides of the cylinder, $\overset{(n)}{T}_i = 0$ on the sides. Since the surfaces are smooth and the ends are parallel to $x_1 = 0$, on the latter we have $\overset{(n)}{T}_2 = \overset{(n)}{T}_3 = 0$. $\overset{(n)}{T}_1$ is odd in x_1. Consequently the right-hand side of (5) becomes

$$2\left\{\frac{1+\nu}{E}\int\limits_{x_1=a} a\overset{(n)}{T}_1\, dS - \frac{\nu}{E}\int\limits_{x_1=a} a\overset{(n)}{T}_1\, dS\right\}.$$

† This statement is only approximately true, as the cylinder will expand laterally under pressure. The value of the integral $\int\limits_{x_1=a} u_1\, dS$ will be underestimated by a fraction of the order of Poisson's ratio times the mean angle which the normal to the rigid surface over its area of contact makes with the axis Ox_1. The result obtained (equation (8)) is exactly true when the displacement u_1 at the ends of the cylinder is explicitly given.

Now the force F_c on either end $= \int\limits_{x_1=a} \overset{(n)}{T}_1\, dS$ by (7). But, by (5),

$$2\int\limits_{x_1=a} u_1\, dS = 2\left\{\frac{1+\nu}{E}\int\limits_{x_1=a} a\overset{(n)}{T}_1\, dS - \frac{\nu}{E}\int\limits_{x_1=a} a\overset{(n)}{T}_1\, dS\right\},$$

or

$$\int\limits_{x_1=a} \overset{(n)}{T}_1\, dS = \frac{E}{a}\int\limits_{x_1=a} u_1\, dS.$$

Therefore the force F_c on either end $= \dfrac{E}{a}\int\limits_{x_1=a} u_1\, dS.$ (8)

If the cylinder is compressed further, let the corresponding displacement be U. Then if v_1 is the displacement of any point on the surface, $v_1 = u_1+U$. By a similar analysis the force on the end is now

$$F = \frac{E}{a}\int\limits_{x_1=a} v_1\, dS = \frac{E}{a}\int\limits_{x_1=a} u_1\, dS + \frac{E}{a}\int\limits_{x_1=a} U\, dS,$$

whence

$$F = F_c + \frac{E}{a}UA, \tag{9}$$

where A is the area of the plane end. Equation (9) implies that once the surfaces are in complete contact with the cylinder, the additional force required for any further compression is equal to that force that would be required if the rigid surfaces were plane.

Before the surfaces establish complete contact with the ends, equation (5) only gives an inequality. The integrand of $\int\limits_{x_1=a} u_1\, dS$ is not known where the end is not in contact. It is less than the value necessary to bring it into contact with the plate.

If known forces symmetrical about $x_1 = 0$ are applied to the sides of the cylinder the above argument is unaltered except that these forces contribute to the second term on the right-hand side of (5).

The final result is

$$\int\limits_{x_1=a} \overset{(n)}{T}_1\, dS = \frac{E}{a}\int\limits_{x_1=a} u_1\, dS + \frac{\nu}{a}\int \left(x_2\overset{(n)}{T}_2 + x_3\overset{(n)}{T}_3\right) dS, \tag{10}$$

the last integral being taken over that part of the curved surface of the cylinder where $x_1 \geqslant 0$.

As a particular example consider two rigid spheres each of radius R_1 compressing a right circular cylinder of radius R_2 and height $2a$. The edges of the cylinder are just in contact with the spheres.

If r is the radius to any point on the surface of the cylinder,

$$-u_1(r) = \frac{R_2^2}{2R_1} - \frac{r^2}{2R_1},\dagger$$

so that
$$-\int\limits_{x_1=a} u_1\,dS = -\int\limits_0^{R_2} \left(\frac{R_2^2}{2R_1} - \frac{r^2}{2R_1}\right) 2\pi r\,dr = -\tfrac{1}{4}\pi \frac{R_2^4}{R_1}.$$

The compressive force on either sphere is, by (8), equal to

$$\frac{\pi E R_2^4}{4aR_1}. \tag{11}$$

3. Applications to plain strain

An equation (12) analogous to (1) exists in plane strain. Consider any cross-section of the body of area S whose normal lies in the direction of plane strain Ox_3. Let the cross-section be bounded by the curve C. Then

$$\int\limits_C u_i n_j\,ds = \int\limits_S \frac{\partial}{\partial x_j} u_i\,dS$$

by Green's lemma in two dimensions; n_i are the direction cosines of the outward normal to C and suffixes take the values 1 and 2 only. Hence,

$$\int\limits_C (u_i n_j + u_j n_i)\,ds = \int\limits_S \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) dS = \int\limits_S e_{ij}\,dS$$
$$= \int\limits_S \left\{\frac{1+\nu}{E} p_{ij} - \frac{\nu(1+\nu)}{E}\delta_{ij} p_{kk}\right\} dS$$

by Hooke's law applied to plane strain. Now

$$\int\limits_C x_i \overset{(n)}{T_j}\,ds = \int\limits_C x_i p_{jk} n_k\,ds = \int\limits_S \frac{\partial}{\partial x_k}(x_i p_{jk})\,dS = \int\limits_S \left(\delta_{ik} p_{jk} + x_i \frac{\partial}{\partial x_k} p_{jk}\right) dS$$
$$= \int\limits_S p_{ij}\,dS + \int\limits_S \rho x_i (f_j - X_j)\,dS$$

by the equations of motion, whence

$$\frac{1}{2}\int\limits_C (u_i n_j + u_j n_i)\,ds = \frac{1+\nu}{E}\int\limits_C x_i \overset{(n)}{T_j}\,ds - \frac{\nu(1+\nu)}{E}\delta_{ij}\int\limits_C x_k \overset{(n)}{T_k}\,dS -$$
$$-\frac{1+\nu}{E}\int\limits_S \rho x_i (f_j - X_j)\,dS + \frac{\nu(1+\nu)}{E}\delta_{ij}\int\limits_S \rho x_k (f_k - X_k)\,dS. \tag{12}$$

† Subject to the limitation explained in the note on p. 381.

For $i = 1, j = 1$ and in the absence of body forces and accelerations, i.e. in equilibrium with no body forces,

$$\int_C u_1 n_1\, ds = \frac{1+\nu}{E} \int_C x_1 \overset{(n)}{T_1}\, ds - \frac{\nu(1+\nu)}{E} \int_C \left(x_1 \overset{(n)}{T_1} + x_2 \overset{(n)}{T_2}\right) ds$$

$$= \frac{1-\nu^2}{E} \int_C x_1 \overset{(n)}{T_1}\, ds - \frac{\nu(1+\nu)}{E} \int_C x_2 \overset{(n)}{T_2}\, ds. \tag{13}$$

As an example of the use of (13), we consider the compression of a rectangle between two smooth rigid curves, which are symmetrical about the line $x_1 = 0$. The sides of the rectangle in contact with the curves are $x_1 = \pm a$. Assuming that contact just exists all along $x_1 = \pm a$, u_1 is known on $x_1 = \pm a$ from the shape of the rigid curves. On $x_1 = a$, $n_1 = +1$; on $x_1 = -a$, $n_1 = -1$; and on the other two sides $n_1 = 0$. Now u_1 is odd in x_1; hence

$$\int_C u_1 n_1\, ds = 2 \int_{x_1=a} u_1\, ds.$$

The compressive force F per unit width $= -\int_{x_1=a} \overset{(n)}{T_1}\, ds$; on C $\overset{(n)}{T_2} = 0$ everywhere. Therefore, from (13),

$$2 \int_{x_1=a} u_1\, ds = \frac{2(1-\nu^2)}{E} a \int_{x_1=a} \overset{(n)}{T_1}\, ds = -\frac{2(1-\nu^2)a}{E} F,$$

so that

$$F = -\frac{E}{(1-\nu^2)a} \int_{x_1=a} u_1\, ds. \tag{14}$$

In particular for two right circular cylinders of equal radius R pressed symmetrically in plane strain into a strip of rectangular cross-section and breadth $2b$, on $x_1 = a$

$$-u_1(x_2) = \frac{b^2}{2R} - \frac{x_2^2}{2R},\dagger$$

$$\int_{-b}^{b} -u_1(x_2)\, dx_2 = \frac{2b^3}{3R},$$

and

$$F = \frac{2Eb^3}{3(1-\nu^2)Ra}. \tag{15}$$

A further application of equation (13) is in the cold rolling of metals. It is used to calculate the compressive force acting on the strip in the regions where it is elastically compressed between the rollers (**3**).

† Subject to the limitation explained in the note on p. 381.

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THE QUARTERLY JOURNAL OF MECHANICS
AND APPLIED MATHEMATICS

VOLUME VI PART 3 SEPTEMBER 1953

CONTENTS

R. V. SOUTHWELL: Some Extensions of 'Rayleigh's Principle' . 257

D. S. SAXON and A. S. CAHN: Modes of Vibration of a Suspended Chain 273

J. W. MILES: Virtual Momentum and Slender Body Theory . 286

S. TOMOTIKA and T. AOI: The Steady Flow of a Viscous Fluid past an Elliptic Cylinder and a Flat Plate at Small Reynolds Numbers 290

J. D. PEARSON: The Transient Motion of Sound Waves in Tubes 313

J. N. HUNT: A Note on Gravity Waves of Finite Amplitude . 336

R. TIFFEN: Generalized Plane Stress Problems in Infinite Elastic Strips 344

E. H. MANSFIELD: Neutral Holes in Plane Sheet—Reinforced Holes which are Elastically Equivalent to the Uncut Sheet . 370

D. R. BLAND: Mean Displacements on the Boundary of an Elastic Solid 379

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257

273

286

290

313

336

344

370

379

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